PRODUCTION THEORY WITH CONVEX LABOR FRICTION: FOUNDATION OF AN OPTIMAL NON-MARKET-CLEARING ECONOMY

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Abstract. This paper provides a general framework for the supply side when a firm employing multiple workers faces convex search friction. It also provides analytical scheme for non-steady states. Convex search friction makes the path outside of unbounded steady states possess greater significance than mere transition, since any level of output can be supported as a steady state equilibrium depending on the state of expectation. The marginal profit value of labor is always strictly positive, which results in persistent excess demand in the labor market as well as excess supply in the goods market. However, the fact that convex search friction makes immediate adjustment of employment suboptimal induces further hiring to depend on coordination of expectation. It raises non-market-clearing equilibrium in terms of long-run without price- nor wage-rigidity in competitive environments, since flexible prices eliminate only temporary disequilibrium but not coordination failure.

1. Introduction

The present paper studies optimal employment policy of a firm operating in a frictional labor market in which friction is representable by a convex vacancy cost function, and studies its implication on employment distribution and market equilibrium. In generalizing a matching model from one-to-one to one-to-many, it is natural, or even necessary, to assume a convex vacancy cost function —a function which literally relates number of job vacancy posting to the cost. The existence of friction make it inevitable to use internal resources to hire workers so that it causes congestion over those resources. Search friction inherently arises from heterogeneity in undocumentable properties of workers or firms. Any friction attributable to documentable properties can be part of friction, but it is easily eliminated with information technology to a negligible level. As such, selection of workers requires tacit knowing by insiders, and therefore, if a firm intends to increase number of hiring, the accompanying cost should exhibit more than linear increase as any kinds of adjustment cost do (Uzawa (1969), Yashiv (2000, 2006)). Yashiv (2000) shows that empirical hiring cost function is highly convex in terms of weighted average of posted vacancy and actual hiring. Moreover, convexity in the vacancy cost function is supported from the fact that it is the only class robust against small perturbations on functional form, the derivative of which is monotone and which does not diverge in equilibrium. Requirement of robustness would be natural, because there is no logically

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deterministic, *a priori* reason that the vacancy cost function must have a particular functional form. For an analysis to be applicable to actual economies, it should be based on robust assumptions.

It will be shown that the marginal wage cost value determined by bargaining is *always* strictly smaller than the marginal production value of labor as far as employment is below the unbounded steady state. On the other hand, the optimal employment policy under the convex vacancy cost function does not allow the path to jump to the unbounded steady state. This is in contrast to the linear vacancy cost case of Smith (1999). Any firm is willing to accommodate all the demand directed to it at any moment. However, investment decision on hiring cost depends on the expected demand in the next moment which is contingent on the action of other firms. For an economy to converge to the unbounded steady state, agents must share dynamically persistent common knowledge that the economy ultimately reaches to that state. Once agents become to believe that demand is already satiated, strategy to make additional employment becomes suboptimal, and thus demand will not actually grow. This brings the same implication as non-market-clearing (NMC) approach by Barro and Grossman (1976) on rational basis with flexible prices. Flexible prices will eliminate temporary disequilibrium but not the state of long-run disequilibrium that arises from the degree of coordination. The model presented in this paper is a generalization of the Mortensen-Pissarides model which assumes that production is undertaken by a pair of a worker and a job. The Mortensen-Pissarides model can be interpreted that it assumes that the "firm" employing multiple workers is decomposable to independent units of jobs. Its assumption of constant vacancy cost is literally hypothesizing a linear vacancy cost function. It assumes that the size of employment in the economy is determined by the entry condition that the value of vacancy equals zero, together with the assumption that each production pair always *successfully* earns constant income. However, it is not guaranteed that the level of employment this condition requires is not so huge that it exceeds the existing population. At the unbounded steady state, we should observe that potential entrepreneurs cease job posting simply because labor market tightness makes the waiting time for arrival of workers too long compared to vacancy cost. However, such behavior can be observed, if any, only in the acme of economic boom. The existence of free public intermediaries and the fact that vacancy cost should decline as market tightness increases enforce the view that the required employment is too high. The decrease of vacancy cost is due to the decrease of applicants that a particular firm receives requiring less cost for interview and selection activities. Also, linear hiring cost implicitly assumes that hiring activities do not consume any internal resources. Namely, the selection process must be trivial so that it simply does not exist or it is outsourced. In the former case, search friction due to heterogeneity of agents will not exist. The latter case implies that all properties related to worker selection must be describable to delegate the selection.

Section 2 summarizes the structure of the model. Section 3 studies how the value of employment is determined. Section 4 studies how the outcome of wage bargaining is actually paid in the form of wages. Non-stationary value functions are solved using the integral equations. Section 5 studies the firm's optimal employment behavior. Section 7 analyzes the behavior on the demand constraint when the constraint is stationary. Section 8 delivers some implication on interest rates, which might resolve the allocation paradox. Section 9 provides some concluding remarks.

2. The Model

2.1. **Firm.** A competitive firm under the presence of search friction in the labor market is considered. The number of firms is continuous with fixed measure one.[1] Namely, one particular firm has continuously small measure. Therefore, the change in the supply of the firm cannot affect aggregate supply, resulting not only in unchanged prices but also in unchanged aggregate income. There are two goods in the economy: output goods and various types of labor. Output goods is taken to be numeraire and labor is heterogeneous with unknown properties so that optimal search behavior of a firm is not trivial. A firm uses multiple workers of potentially different types. Types of workers can be categorized into two classes, declarable types and non-declarable ones. The former is the properties that can be prescribed as hiring requirements, such as possession of driver's license and academic background. The latter is the attributes that cannot be documented. Therefore, they are only revealed after having an interview with workers. Personality or suitability to particular corporate culture falls in this category. The notion of quality of match is also covered by this concept as far as it is revealed immediately. Non-declarable types are assumed to be matching-specific. Worker types are expressed by a combination of declarable and non-declarable types, therefore let bundle (i, j) denote a worker type with declarable type $i \in \{1, \ldots, L\}$ and non-declarable type $j \in \{1, \ldots, M_i\}$. Different types are clearly distinguished from each other. Workers are assumed to be unable to choose their types to abstract the effect of education and training. Same type of workers are homogeneous. Production function of a firm is given by $f(\boldsymbol{l})$ where $\boldsymbol{l} = (\boldsymbol{l}_1, \ldots, \boldsymbol{l}_L)$ and $\boldsymbol{l}_i = (l_{i1}, \ldots, l_{iM_i})$ are vectors of employment such that l_{ij} is employment of (i, j)-type worker, $\partial f/\partial l_{ij} > 0$ and f is concave. We also assume Inada condition around the origin: $\partial f/\partial l_{ij} \to +\infty$ as $l_{ij} \to 0$.

Since labor market is frictional, a firm cannot adjust employment stock directly. It can only adjust inflow to and outflow from the stock. For the inflow, the firm decides how much internal resources to spend on recruiting workers in the labor market to adjust employment. After the choice of the level of recruiting activities $\boldsymbol{m} = (m_1, \ldots, m_L)$ for each declarable type, it would observe a variety of applicants to arrive stochastically.[2] A matching session proceeds in a way that firms post job advertisement first and workers apply to a preferred job. Such a matching mechanism

[1]It should be emphasized that this is *not* assuming away free-entry to get the results obtained in this paper. The reason that this condition is required is that, 1) a vacancy cost function with no fixed cost, 2) variable measure of firms and 3) the production economy simply cannot coexist. Once we allow variable measure of firms, infinite number of firms employing no workers is created, leading to zero production (not indeterminate, see arguments below). To describe production economy, either 1) or 2) must be discarded.
The reason that the vacancy cost function κ such that $\kappa(0) = 0$ is incompatible with variable measure of firms bounded by no-entry condition $J(\mathbf{0}, y) = 0$ is as follows (please refer to later pages for definitions). If $y = 0$, then $J(\mathbf{0}, y) = 0$ holds with no firm entry. On the other hand, if $y > 0$, firm entry must continue until $J(\mathbf{0}, y) = 0$ is restored with $y = 0$ (adjustment through θ alone cannot bring $J = 0$ since, for any high value of θ, the entrant firm can set arbitrarily small m). Thus, no production equilibrium can be supported without fixed costs.
The other strategy to build a model for production economy would be to throw away 1), namely to assume that there exists $\underline{\kappa} > 0$ such that $\kappa(0) = \underline{\kappa}$. It implies that the no-entry condition holds with $y > 0$. Since $\partial J(t)/\partial l_{ij}(t) > 0$, it makes new entrants give up, whereas existing firms are willing to operate. Hereby, simply assuming $\kappa'(0) > 0$ will not suffice, because it will make all firms shut down by optimality condition (5.6) so far as entrants do not want to enter. Since the implication of such an alternative model does not significantly differ from the present model, the simpler model is adopted. However, the emergence of the distribution of firm size is abstracted by doing so, which would have arisen in the alternative model by historical movement of y.

[2]By equation (2.1), m_i is directly related to the increase of labor force. It is labeled as "level of recruiting activities" instead of "number of job vacancies" to abstract the strategic behavior to announce more job posts than actually wanted.

would be a natural equilibrium when most of characteristics of firms relevant to matching are declarable whereas those of workers are non-declarable. It results in the same outcome as random matching, so that the probability that a firm receives applications per job posting is a decreasing function of the *VU* ratio in the labor market of declarable type i, θ_i. It is denoted by $\psi(\theta_i) : \mathbb{R}_+ \to \mathbb{R}_+$.[3] On the other hand, non-declarable type is matching specific and non-declarable type j emerges with probability $g_{ij} \in (0, 1)$ among declarable type i where $\sum_j g_{ij} = 1$. Therefore, if a firm exerts recruiting effort m_i on declarable type i in the labor market, it will receive $g_{ij}\,\psi(\theta_i)\,m_i$ applications from type (i, j) worker.

On the other hand, there are two factors which affect outflow from the employment stock. One is an uncontrollable leave of workers.[4] This natural separation of type (i, j) worker at time t is a Poisson arrival with parameter $\sigma_{ij}(t) > 0$.[5] The other factor is intentional dismiss by the firm. Dismissal of type (i, j) worker is denoted by $x_{ij} \in [0, X]$ where $X \in (0, \infty)$ is a sufficiently large number and can be interpreted as a physical boundary of adjustment speed of employment downward. Note that the firm can specify the worker type to dismiss since the non-declarable property of a worker is revealed during the employment period.

Now, the firm can control the time derivative of type (i, j) employment using m_i and x_{ij} so that

$$\dot{l}_{ij} = g_{ij}\,\psi(\theta_i)\,m_i - \sigma_{ij} l_{ij} - x_{ij} \qquad \forall i = 1, \ldots, L,\ \forall j = 1, \ldots, M_i. \tag{2.1}$$

Notation $\phi_{ij}(t) := g_{ij}(t)\,\psi(\theta_i(t))$ is sometimes used for simplicity.

Job posting is assumed to be costly. Smith (1999) assumed a linear vacancy cost function, so that a firm employs all necessary workers to reach to the steady state in the first period, and then it maintains the steady state forever. With this carefully arranged setup, adjustment process to the steady state is virtually abstracted but at the same time it omits robustness against small perturbations on functional form. To restore robustness, we are induced to assume a convex vacancy cost function following a minimalist principle, which has the property that the derivative is monotone and also the implied equilibrium outcome does not diverge. It would be a natural assumption from the viewpoint that a vacancy cost function should be regarded as an adjustment cost function. In practical applications, the cost should be interpreted to include the cost for orientation, training and deterioration of productivity that arises from on-the-job training and inexperience of new workers, as well as the cost necessary for actual recruiting. Modifying the functional form in this way definitely complicates the analysis compared to a linear case, but it should be emphasized that such a model brings significantly *different* macroeconomic implications. We denote the vacancy cost function of declarable type i by $\kappa_i(m_i, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ where $\kappa_i \geq 0$, $\partial\kappa_i/\partial m_i \geq 0$, $\partial^2\kappa/\partial m_i^2 > 0$, $\kappa_i(0, t) = \partial\kappa_i/\partial m_i(0) = 0$, and the second argument will be sometimes omitted for simple notation, i.e. $\kappa_i(\cdot) := \kappa_i(\cdot, t)$. The time-dependency of κ_i allows it to depend on θ_i. Another cost for a firm is

[3] This is a special case of Yokota (2004) so that the quality threshold of a firm to decline an applicant is set to zero to conform the current problem setting. In this paper, rejection is detached to the choice of x_{ij}.

[4] Since it turns out later that the value of employment status is always greater than the value of unemployment status, this shock is not only external for firms but also for workers.

[5] It may be more natural to assume that the quality of a match gradually turns out on the job as in Jovanovic (1979). However, we abstract internal working of separation.

wage payment to each type of workers. Wage payment is determined by bargaining between a firm and workers to share their rent that arises from historical advantages that an already-formed coalition possesses. Wage rate will be shown to be a function of employment. Therefore, the firm decides the amount of employment knowing the wage schedule it faces. We denote the wage function of type (i, j) by $w_{ij}(\boldsymbol{l})$.

On the demand side, consumers demand outputs based on information about current assets, current income and *the state of expectation on series of future income* which is influenced by the degree of coordination among agents. The expectation on intertemporal income holds both on and off equilibrium paths. In the present model, temporary equilibrium is brought by adjustment in the interest rate. Temporary supply is fixed at any given instance, therefore interest rate adjusts temporary demand to meet it affecting intertemporal relative prices. The model also allows for analysis when disturbances are added to the adjustment in interest rates in the goods market. In such a case, temporary disequilibrium arises in addition to long-run disequilibrium. Another such instance is the case of perishable goods since it physically prohibits intertemporal consumption smoothing by consumers. Let us denote by $y(t)$ the "demand density" defined by

$$y(t) = f(\boldsymbol{l}(t), n) + \left(D(r, \tilde{y}, t) - \int_0^1 f(\boldsymbol{l}(t), n)\, dn \right) \tag{2.2}$$

and

$$\dot{y}(t) = \dot{D}(r, \tilde{y}, t) \tag{2.3}$$

where we explicitly denote the production function of firm with index n by $f(\boldsymbol{l}, n)$ assuming integrability in n, and $D(r, \tilde{y}, t)$ is the total demand for outputs as of time t which is affected by interest rate $r(\tau)$ $(t \leq \tau < \infty)$ and income stream $\tilde{y}(\tau)$ $(t \leq \tau < \infty)$. It also includes "queue" of demand which has not met by previous supply. Equation (2.3) and the second term of equation (2.2) impose homogeneity among firms in terms of their future prospectives, since they imply that additional demands are equally distributed among them. Modification of these terms will introduce heterogeneity in their, say, marketing power and others. Thus, y should be understood as rational expectation among market participants on prospectives of each firm. Since any firms cannot directly affect r, y is given for any particular firm under rational expectation.[6] Normalizing the price of output to one, the instantaneous profit of the firm π is given by

$$\pi = \min\{y, f(\boldsymbol{l})\} - w(\boldsymbol{l}) \cdot \boldsymbol{l} - \sum_{i=1}^{L} \kappa_i(m_i).$$

[6]As Arrow (1959) argued that the economy will show evidences of monopoly and monopsony in any state of disequilibrium, pricing strategy must be examined carefully here. Firms can potentially use their profits to lower nominal prices and expand their market share permanently. However, such a move will be retaliated by other firms, resulting in failure of the original intention of market share extension. If demand is elastic in interest rate, it increases temporary market production and lowers interest rate, but reduces future demand. Such a strategy costs more than proportionate hiring costs but shifts demand from future to today. We assume away such a strategic move in price settings and the firm simply takes the bargaining outcome of real wages. When demand has zero elasticity, there is no incentive to manipulate prices.

When $\min\{y, f(\boldsymbol{l})\} = y$, as far as there is no reason for labor hoarding, increasing $\boldsymbol{l}$ to make $f(\boldsymbol{l}) > y$ is obviously suboptimal. Since the above formulation introduces indifferentiability in π that makes succeeding analysis difficult. In stead of handling the above formulation directly, we are going to analyze an "approximate" problem that the firm maximizes profit $\pi = f(\boldsymbol{l}) - w(\boldsymbol{l}) \cdot \boldsymbol{l} - \sum_{i=1}^{L} \kappa_i(m_i)$ under the constraint $f(\boldsymbol{l}) \leq y$. It ignores cases in which a firm hoards labor facing *temporary* demand recession, however their long-run outcome should be quite similar. If it is expected that labor hoarding never occurs in future, both outcomes exactly coincide.[7] Again, for simple notation, the following notation is sometimes used: $c_{ij}(\boldsymbol{l}) := w_{ij}(\boldsymbol{l})\, l_{ij}$ for any (i, j).

2.2. **Workers and consumers.** Workers are in either state, employed or unemployed. An unemployed worker of type $(i, \cdot)$ at time t receives instantaneous unemployment benefit $b_i(t)$. An employed worker of type (i, j) will be paid instantaneous wage $w_{ij}(t)$. The value of type-$(i, \cdot)$ unemployment at time t is denoted by $U_i(t)$ and the value of type-(i, j) employment at time t is denote by $E_{ij}(t)$. Matching sessions between job-seekers and vacancies open at any moment. Matching probability of an unemployed worker is given by $\mu_i(t)$. It is in general influenced by $\theta_i(t)$ but we suppress its explicit notation. Matching sessions are time-consuming and the length is random. Agents cannot attend other matching sessions simultaneously while he is engaged in a particular session.[8] When an unemployed worker of declarable type i is successfully matched to find out his undeclarable type is j, he shifts to the employment status of type (i, j). Namely, on success, he receives capital gain $E_{ij}(t) - U_i(t)$. Assuming workers are risk neutral, the Bellman equation for unemployment status is

$$r(t)\, U_i(t) = b_i(t) + \mu_i(t)\, \mathsf{E}_j \left[E_{ij}(t) - U_i(t)\right] + \dot{U}_i(t), \qquad \forall i \tag{2.4}$$

where r is interest rate and $\mathsf{E}_{\mathrm{j}} \left[E_{ij}(t) - U_i(t)\right] := \sum_{j=1}^{M_i} g_{ij}(t)\, E_{ij}(t) - U_i(t)$. Similarly, the employment value of type (i, j) is given by

$$r(t)\, E_{ij}(t) = w_{ij}(t) - \tilde{\sigma}_{ij}(t) \left[E_{ij}(t) - U_i(t)\right] + \dot{E}_{ij}(t), \qquad \forall i, j \tag{2.5}$$

where $\tilde{\sigma}_{ij}(t) := \sigma_{ij}(t) + x_{ij}(t)/l_{ij}(t)$ is separation probability for a worker.

3. Bargaining over Coalitional Rent

When there exists friction in the labor market, pseudo-rent arises in an existing firm-workers group. The rent comes from the fact that any firms or workers who have not formed a group yet must enter a costly process of search. It makes a room for bargaining over the rent between a firm and workers who have already formed a group. Therefore, production should be regarded as coalitionally undertaken by the going concern and present employees, and distribution of payoffs is made through bargaining taking into account future renewal of players.

[7]If unintended inventory investment is included in the definition of y as in that of national accounts, this problem will not arise.
[8]If matching sessions open continuously and end instantaneously, all agents on one side of the labor market will be matched immediately almost surely.

Coalition of a firm and each type of workers with measure l_{ij} will get intertemporal payoff $\bar{F}$ which is the value of

$$F\big(\boldsymbol{l}(t)\big) = \int_t^\infty \left[f(\boldsymbol{l}^*(\xi)) - \sum_{i=1}^{L} \kappa_i(m_i^*(\xi)) \right] e^{-\int_t^\xi r(\tau)d\tau} d\xi$$

where $\boldsymbol{l}^*$ and m_i^* are *on the optimal path for the firm* in the concomitant problem defined in section 5. In this coalition, bargaining is made among continuously many players. It will be found that the imputation to give workers just the half of their marginal gain, i.e.

$$E_{ij} = \frac{1}{2}U_i + \frac{1}{2}\frac{\partial F}{\partial l_{ij}}$$

is supported as a bargaining solution by Shapley value and additionally by nucleolus if F shows concavity.[9] For the moment, to clarify conditions characterizing the solution, argument proceeds with a general setup that wage bargaining should satisfy.

Let Ω be a set of all players. Players are partitioned by groups S_i ($i = 0, 1, \ldots, M$; $M \in \mathbb{N}$) such that $\cup_{i=1}^N S_i = \Omega$ and $\cap_{i=1}^N S_i = \emptyset$. Each group consists of $N_i \in \mathbb{N} \cup \{\infty\}$ players. The j-th player in group S_i is denoted by $s_i(j)$ ($j = 1, \ldots, N_i$). $s_i(j)$ has measure dl_i for all j and there exists a fixed number $l_i \in \mathbb{R}_+$ such that $N_i\, dl_i = l_i$ for all i. Characteristic function is denoted by $v : 2^\Omega \to \mathbb{R}$. We require following assumptions:

(1) (Essential game) The game is essential, i.e. $v(\Omega) > \sum_{s\in\Omega} v(\{s\})$.
(2) (Anonymity) Players in the same group are anonymous, i.e. for any S and j, $v\big(S \cup \{s_i(j)\}\big) - v(S)$ is common.
(3) (Indispensability) Missing groups make coalition unproductive, i.e. $v(S) = \sum_{i,j\in S} v(\{s_i(j)\})$ if there exists i such that $S \cap S_i = \emptyset$.
(4) (Existence of a non-degenerate player) S_0 is a special group which consists of only one player, i.e. $N_0 \equiv 1$.

Refer to this game with symbol $\mho_{\boldsymbol{N}}^1(\Omega, v)$ where $\boldsymbol{N} := (N_0, \ldots, N_M)$. Next, define a more specific game within the class of $\mho_{\boldsymbol{N}}^1(\Omega, v)$ which possesses essential concavity as an additional assumption. It requires concavity only for coalitions with more than two players so that essentiality of the game will not be lost.[10] Namely, we put the following additional property:

(5) (Essential concavity) The game is *essentially concave*, i.e. its characteristic function v has the property that, if $S, T \subseteq \Omega$ satisfies $\emptyset \subset S \subset T$, then

$$v(S \cup \{s\}) - v(S) \geq v(T \cup \{s\}) - v(T)$$

for any $s \in \Omega \setminus T$.

[9]We require a singleton solution to proceed with the model. Pissarides (1985) assumed that, in the case that production is undertaken by a pair of a firm and a worker, they divide the rent by a Nash bargaining solution. There is an option to generalize it adopting n-player Nash equilibrium. However, since the present model contains significant asymmetry between a firm and workers, it seems more natural to take coalitional rationality into account.

[10]A globally concave game always violates zero-additivity, thus formation of non-trivial coalitions cannot be expected.

Refer to this game with symbol $\eth^2_{\boldsymbol{N}}(\Omega, v)$. Finally, define an even more specific game $\eth^3_{\boldsymbol{N}}(\Omega, v)$ which is directly related to our problem. Players are grouped by categories which are classified in two dimensions so that any player belongs to $\tilde{S}_{ij}$ for $i = 0, \ldots, L$ and $j = 1, \ldots, M_i$ where $\bigcup_{i=0}^{L} \bigcup_{j=1}^{M_i} \tilde{S}_{ij} = \Omega$ and $\tilde{S}_{ij} \cap \tilde{S}_{i'j'} = \emptyset$ for any $(i, j) \neq (i', j')$. Characteristic function is defined by

$$v(S) = \begin{cases} \sum_{i\in S} U_i & \text{if there exists } i \text{ such that } S \cap \tilde{S}_{ij} = \emptyset \\ F(\tilde{l}_{11}, \ldots, \tilde{l}_{LM_L}) & \text{otherwise} \end{cases} \tag{3.1}$$

where $U_i \geq 0$ for all i, $\tilde{l}_{ij} := \left\| S \cap \tilde{S}_{ij} \right\|$ and F is increasing and concave.[11] Obviously, this is a special case of $\eth^2_{\boldsymbol{N}}(\Omega, v)$ with $\tilde{S}_{ij} = S_{\sum_{k=0}^{i-1} M_k + j - 1}$. However, we will not utilize the above additional specifications to characterize solution concepts. They are only used to relate the results obtained in this section to the rest of the paper.

Our objective is to obtain a bargaining solution of the above games when $\boldsymbol{N} \to \infty'$ where $\infty' := (\infty, \ldots, \infty) \in (\mathbb{R} \cup \{\infty\})^M$ keeping l_i fixed for all $i > 0$. Note that, by doing so, the firm $s_0(1)$ keeps discrete influence on coalitional payoff. The property that workers get only partial contribution depends on the assumption that players in S_0 does not degenerate, rather than the particular value assumption $N_0 \equiv 1$. Also, note that concavity of v and F is sufficient to hold *only from below* at Ω and $\boldsymbol{l}$, respectively, i.e. the concavity need not hold for supersets of Ω or any $\hat{\boldsymbol{l}} \geq \boldsymbol{l}$ with $\hat{l}_{ij} > l_{ij}$ for some i, j. This fact will be used in section 6. Denote the density imputation to player s by $\iota(s)$, i.e. imputation of player s with measure dl becomes $\iota(s)\, dl$.

Theorem 1. *In $\eth^1_{\infty'}(\Omega, v)$, the imputation to allocate*

$$\iota(s_i(j))\, dl_i = \frac{1}{2} v\Big(\{s_i(j)\}\Big) + \frac{1}{2}\Big[v(\Omega) - v\Big(\Omega \setminus \{s_i(j)\}\Big)\Big] \tag{3.2}$$

to any workers of type (i, j) is supported by Shapley value.

Proof. Choose a player $s_{\hat{\imath}}(\hat{\jmath})$ for some $\hat{\imath}$ and $\hat{\jmath}$. Consider any coalition S such that $s_{\hat{\imath}}(\hat{\jmath}) \in S$ containing n_i players from group S_i such that $n_i \geq 0$ and $n_{\hat{\imath}} \geq 1$. The contribution of $s_{\hat{\imath}}(\hat{\jmath})$ to coalition S is $v(\{s_{\hat{\imath}}(\hat{\jmath})\})$ if there exists i such that $S \cap S_i = \emptyset$ from the indispensability assumption. In other cases, it is $v(S) - v(S \setminus \{s_{\hat{\imath}}(\hat{\jmath})\})$. The Shapley's weight $\gamma(S)$ for the contribution of $s_{\hat{\imath}}(\hat{\jmath})$ to coalition S is given by

$$\begin{aligned} \gamma(S) &= \frac{(\sum_{i=0}^{M} n_i - 1)!(\sum_{i=0}^{M} N_i - \sum_{i=0}^{M} n_i)!}{(\sum_{i=0}^{M} N_i)!} \\ &= \left(\sum_{i=0}^{M} N_i\right)^{-1} \binom{\sum_{i=0}^{M} N_i - 1}{\sum_{i=0}^{M} n_i - 1}^{-1}. \end{aligned}$$

Without loss of generality, let us assume $\hat{\imath} = 1$ below for concise notations. From the anonymity assumption, any S with same $(n_0, \ldots, n_M)$ has the same $\gamma(S)$. The number of cases to form coalition S containing $s_{\hat{\imath}}(\hat{\jmath}) = s_1(\hat{\jmath})$

[11]The assumption of global concavity in F is actually asking too much than necessary. It is sufficient if F satisfies $F(\tilde{\boldsymbol{l}}) \leq F(\boldsymbol{l}) - \partial F(\boldsymbol{l})/\partial \boldsymbol{l} \cdot (\boldsymbol{l} - \tilde{\boldsymbol{l}})$, $\forall \tilde{\boldsymbol{l}} \leq \boldsymbol{l}$ for the current level of employment $\boldsymbol{l}$.

with same $(n_0, \ldots, n_M)$ is given by

$$\binom{N_0}{n_0} \cdot \binom{N_1 - 1}{n_1 - 1} \cdot \binom{N_2}{n_2} \cdots \binom{N_M}{n_M}.$$

Then, Shapley value is given by

$$\iota(s_1(\jmath))\, dl_1 = \left(\sum_{\{S : \prod_{i=0}^M n_i = 0\}} \Gamma(S) \right) v\big(\{s_1(\jmath)\}\big) + \sum_{\{S : \prod_{i=0}^M n_i \geq 1\}} \Gamma(S) \left[v(S) - v\big(S \setminus \{s_1(\jmath)\}\big) \right] \tag{3.3}$$

where

$$\Gamma(S) := \gamma(S) \cdot \binom{N_0}{n_0} \cdot \binom{N_1 - 1}{n_1 - 1} \cdot \binom{N_2}{n_2} \cdots \binom{N_M}{n_M} = \frac{\binom{N_0}{n_0} \cdot \binom{N_1 - 1}{n_1 - 1} \cdot \binom{N_2}{n_2} \cdots \binom{N_M}{n_M}}{\left(\sum_{i=0}^M N_i \right) \binom{\sum_{i=0}^M N_i - 1}{\sum_{i=0}^M n_i - 1}}.$$

Proposition 20 in Appendix A show that coefficient $\Gamma(S)$ is a probability mass function such that $\Gamma(S) = \Upsilon(n_0, n_1 - 1, n_2, \ldots, n_M; N_0, N_1 - 1, N_2, \ldots, N_M)$ where distribution Υ is defined in Appendix A. Note that the distribution possesses point symmetry $\Upsilon(n_1, \ldots, n_M; \zeta_1, \ldots, \zeta_M) = \Upsilon(\zeta_1 - n_1, \ldots, \zeta_M - n_M; \zeta_1, \ldots, \zeta_M)$. Using these two facts,

$$\sum_{\{S : n_0 = 0\}} \Gamma(S) = \sum_{\{S : n_0 = 1\}} \Gamma(S)$$

Now, in either case of $n_0 = 0, 1$, $\sum_{\{S : \prod_{i=1}^M n_i = 0\}} \Gamma(S) \to 0$ as $N_i \to \infty$ for all $i = 1, \ldots, M$. $\sum_{\{S : \prod_{i=1}^M n_i = 0\}} \Gamma(S)$ can be written as

$$\begin{aligned} \sum_{\{S : \prod_{i=1}^M n_i = 0\}} \Gamma(S) &= \frac{\prod_{\{i : n_i = 0, n_1 = 1\}} \binom{N_i}{0}}{\sum_{i=0}^M N_i} \sum_{n_i} \frac{\prod_{\{i : n_i \geq 1, n_1 \geq 2\}} \binom{N_i}{n_i}}{\binom{\sum_{i=0}^M N_i - 1}{\sum_{i=0}^M n_i - 1}} \\ &= \frac{1}{\sum_{i=0}^M N_i} \sum_{n_i} \frac{\prod_{\{i : n_i \geq 1, n_1 \geq 2\}} \binom{N_i}{n_i}}{\binom{\sum_{i=0}^M N_i - 1}{\sum_{i=0}^M n_i - 1}} \end{aligned}$$

where the right hand side converges to zero as $N_i \to \infty$. Therefore,

$$\sum_{\{S : \prod_{i=0}^M n_i = 0\}} \Gamma(S) \approx \sum_{\{S : n_0 = 0\}} \Gamma(S) = \sum_{\{S : n_0 = 1\}} \Gamma(S) \approx \sum_{\{S : \prod_{i=0}^M n_i \geq 1\}} \Gamma(S) \approx \frac{1}{2}.$$

It shows the coefficient of $v\big(\{s_{\imath}(\jmath)\}\big)\, dl_{\imath}$ in (3.3) converges to 1/2 as $\boldsymbol{N} \to \infty'$. From Proposition 21 in Appendix A, we obtain

$$\iota(s_i(j))\, dl_i = \frac{1}{2} v\big(\{s_i(j)\}\big) + \frac{1}{2} \left[v(\Omega) - v\big(\Omega \setminus \{s_i(j)\}\big) \right]$$

for all i, j. □

The above derivation critically depends on the indispensability and the existence assumption of a non-degenerate player, which enable for the firm to keep discrete influence on payoffs whereas that of individual workers becomes negligible as $\boldsymbol{N} \to \infty'$. On the other hand, characterizing bargaining solution as nucleolus requires an additional assumption that the game should be essentially concave. At the outset, the following lemma shows that core is non-empty if and only if the production process is more productive for the last marginal worker than unemployment in terms of value.

Lemma 2. $\mathfrak{D}^1_{\mathbf{N}}(v,\Omega)$ *has non-empty core if it is zero-additive. So does* $\mathfrak{D}^2_{\mathbf{N}}(v,\Omega)$ *if and only if*

$$v\big(\Omega\big) - v\big(\Omega \setminus \{s_i(j)\}\big) \geq v\big(\{s_i(j)\}\big) \tag{3.4}$$

for all i, j. *In* $\mathfrak{D}^3_{\mathbf{N}}(v,\Omega)$, *the condition* (3.4) *is replaced by* $\partial F/\partial l_{ij} \geq U_i$.

Proof. We start from the necessary condition of $\mathfrak{D}^1$. Consider imputation such that any $s \in \Omega \setminus S_0$ is allocated by $\iota(s) = v(\{s\})$ and player $s_0(1)$ is allocated by $\iota(s_0(1)) = v(\Omega) - \sum_{s\in\Omega\setminus S_0} v(s)$. This is feasible by essentiality of the game. Obviously, any S such that $s_0(1) \notin S$ satisfies coalitional rationality since $\sum_{s\in S} \iota(s) \geq v(S) = \sum_{s\in S} v(\{s\})$. So does any coalition S such that $s_0(1) \in S$ since its imputation yields $\sum_{s\in S} \iota(s) = v(\Omega) - \sum_{s\notin S} v(s) \geq v(S)$ by zero-additivity. which implies that this imputation is located in core. If (3.4) holds for $\mathfrak{D}^2$, zero-additivity holds from the essential concavity, which shows that (3.4) is a necessary condition for $\mathfrak{D}^2$. The case for $\mathfrak{D}^3$ is direct from this since $\mathfrak{D}^3$ is a special case of $\mathfrak{D}^2$.

To show (3.4) is a sufficient condition for $\mathfrak{D}^2$, suppose $v(\Omega) - v\big(\Omega \setminus \{s_i(j)\}\big) < v\big(\{s_i(j)\}\big)$ for some i, j. The individual rationality of $s_i(j)$ requires $\iota\big(s_i(j)\big) \geq v\big(\{s_i(j)\}\big)$. Also, coalition of the rest requires $\sum_{s\in\Omega\setminus\{s_i(j)\}} \iota\big(s\big) \geq v\big(\Omega \setminus \{s_i(j)\}\big)$, which implies $\iota\big(s_i(j)\big) \leq v(\Omega) - v\big(\Omega \setminus \{s_i(j)\}\big) < v\big(\{s_i(j)\}\big)$. These two equations are not satisfied at the same time, thus core is empty. It shows zero-additivity is also sufficient. The result for $\mathfrak{D}^3$ is derived from this. □

Following the context of our model in which workers and the firm are all rational in participating in production, the bargaining solution must be in core. Otherwise, at least one player will leave the coalition, which implies that the current coalition is not actually on the optimal path. The above lemma means that the problem can be restricted to the case of $\partial F/\partial l_{ij} \geq U_i$ on the optimal path.

Lemma 3. *If game* (Ω, v) *is essentially concave in which players are partitioned by groups such that* $\Omega = \bigcup_{i=1}^{M} S_i$ *and* $\bigcap_{i=1}^{M} S_i = \emptyset$, *then for any* $S, T \subseteq \Omega$ *such that* $S \subset T$, *the following inequality holds.*

$$v(T) - v(T \setminus S) \geq \sum_{i=1}^{n} \|S \cap S_i\| \left[v(T) - v\big(T \setminus \{s_i(j)\}\big)\right]$$

Proof. See Appendix B. □

This lemma is analogous to the property of an ordinary concave function: $f(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n) \leq f_{x_1}\Delta x_1 + \cdots + f_{x_n}\Delta x_n$ in which each axis corresponds to $\|S \cap S_i\|$. It is required to derive nucleolus of the game.

Theorem 4. *In* $\mathfrak{D}^2_{\boldsymbol{N}}(\Omega, v)$ *for any* $\boldsymbol{N}$, *the imputation* (3.2) *is supported by nucleolus.*[12]

Proof. The proof starts from the following lemma.

[12]This result coincides with Stole and Zwiebel (1996) by extending its result to a case of infinite number of agents.

Lemma 5. *Consider a coalitional game* $\eth^2_{\boldsymbol{N}}(\Omega, v)$ *for given* $\boldsymbol{N}$ *in which* $v(\Omega) - v\big(\Omega \setminus \{s_i(j)\}\big) \geq v\big(\{s_i(j)\}\big)$ *holds for any* i, j. *Then, in any max-reduced game of* $\eth^2_{\boldsymbol{N}}(\Omega, v)$, *the least core* $\Gamma(\varepsilon_n)$ *is characterized by excess*

$$\varepsilon_n = -\frac{1}{2}\left[v(\Omega_n) - v\big(\Omega_n \setminus \{s_{\hat{\imath}}(j)\}\big) - v\big(\{s_{\hat{\imath}}(j)\}\big)\right] \tag{3.5}$$

where Ω_n *is a set of players in the n-th reduced game and* $\hat{\imath} = \arg\min_i v(\Omega_n) - v\big(\Omega_n \setminus \{s_i(j)\}\big) - v\big(\{s_i(j)\}\big)$ *in which* j *can be arbitrary by anonymity.*

Proof of Lemma 5. In the n-th reduced game, characteristic function is given by

$$v(S) = \begin{cases} v(\Omega \setminus \Omega_n \cup S) - \sum_{s \in \Omega \setminus \Omega_n} \iota(s) & \text{if } s_0(1) \in S \\ \sum_{s \in S} v\big(\{s\}\big) & \text{if } s_0(1) \notin S \end{cases} \tag{3.6}$$

for any $S \subseteq \Omega$.

Consider imputation in the ε-core for given excess ε_n. Individual rationality with excess ε requires $\iota(s_i(j))$ to be

$$\iota(s_i(j)) \geq v(s_i(j)) - \varepsilon_n \tag{3.7}$$

for all i, j. On the other hand, coalitional rationality with excess ε_n of the complement of the above, i.e. $\Omega_n \setminus \{s_i(j)\}$, requires $\sum_{s \in \Omega_n \setminus \{s_i(j)\}} \iota(s) \geq v\big(\Omega_n \setminus \{s_i(j)\}\big) - \varepsilon_n$. Since total rationality implies $\iota(s_i(j)) = v(\Omega_n) - \sum_{s \in \Omega_n \setminus \{s_i(j)\}} \iota(s)$, it leads to

$$\iota(s_i(j)) \leq v(\Omega_n) - v\big(\Omega_n \setminus \{s_i(j)\}\big) + \varepsilon_n \tag{3.8}$$

In the payoff space $\{X \in \mathbb{R}^{\sum_i N_i}\}$ where $X := \big(\iota(s_1(1)), \ldots, \iota(s_M(N_M))\big)$, consider a domain which satisfies coalitional rationality of player set S and its complement $\Omega_n \setminus S$ on simplex manifold $\Delta := \{X \in \mathbb{R}^{\sum_i N_i} : \sum_{s \in \Omega_n} \iota(s) = v(\Omega_n)\}$ to satisfy total rationality and denote it by $\mathsf{B}(S, \varepsilon_n)$. Without loss of generality, $s_0(1) \notin S$ can be assumed by symmetry. Then, ε-core is obtained by finding out $\min_\varepsilon\{\varepsilon : \bigcap_{S \in 2^{\Omega_n}} B(S, \varepsilon_n) \neq \emptyset\}$. Generally, $\mathsf{B}(S, \varepsilon_n)$ has the form

$$\mathsf{B}(S, \varepsilon_n) = \left\{X \in \Delta : v(S) - \varepsilon_n \leq \sum_{s \in S} \iota(s) \leq v(\Omega_n) - v(\Omega_n \setminus S) + \varepsilon_n\right\}. \tag{3.9}$$

From (3.7) and (3.8), $\mathsf{B}\big(\{s_i(j)\}, \varepsilon_n\big)$ becomes

$$\mathsf{B}\big(\{s_i(j)\}, \varepsilon_n\big) = \left\{X \in \Delta : v(s_i(j)) - \varepsilon \leq \iota(s_i(j)) \leq v(\Omega_n) - v\big(\Omega_n \setminus \{s_i(j)\}\big) + \varepsilon\right\} \tag{3.10}$$

and therefore

$$\bigcap_{s \in \Omega_n} \mathsf{B}\big(\{s\}, \varepsilon_n\big) = \left\{X \in \Delta : \sum_{s \in \Omega_n} v(\{s\}) - \left(\sum_{i=1}^{M} N_i\right)\varepsilon \leq \sum_{s \in \Omega_n} \iota(s) \leq \sum_{s \in \Omega_n} [v(\Omega_n) - v(\Omega_n \setminus \{s\})] + \left(\sum_{i=1}^{M} N_i\right)\varepsilon\right\} \tag{3.11}$$

Since ε forms least core, it is chosen to make $\mathsf{B}\Big(\{s_i(j)\}, \varepsilon_n\Big)$ non-empty. We are going to show $\min_{\varepsilon_n}\{\varepsilon_n : \bigcap_{S\in 2^{\Omega_n}} B(S, \varepsilon_n) \neq \emptyset\} = \min_{\varepsilon_n}\{\varepsilon_n : \bigcap_{s\in\Omega_n} \mathsf{B}(\{s\}, \varepsilon_n) \neq \emptyset\}$.

From Lemma 2, $\varepsilon_n \leq 0$. Therefore,

$$v(S) - \varepsilon_n \leq v(S) - m\varepsilon_n \leq \sum_{s\in S} v(\{s\}) - m\varepsilon_n \tag{3.12}$$

for any $m \in \mathbb{N}$. On the other hand, from Lemma 3,

$$\begin{aligned} v(\Omega_n) - v(\Omega_n \setminus S) + \varepsilon_n &\geq v(\Omega_n) - v(\Omega_n \setminus S) + m\varepsilon_n \\ &\geq \sum_{s\in S}\Big[v(\Omega_n) - v(\Omega_n \setminus \{s_i(j)\})\Big] + m\varepsilon_n \end{aligned} \tag{3.13}$$

holds for any $m \in \mathbb{N}$. From (3.12) and (3.13), (3.9) and (3.11) imply, for any $S \in 2^{\Omega_n}$, $B(S, \varepsilon) \supseteq \bigcap_{s\in\Omega_n} \mathsf{B}(\{s\}, \varepsilon_n)$, from which $\min_{\varepsilon_n}\{\varepsilon_n : \bigcap_{S\in 2^{\Omega_n}} B(S, \varepsilon_n) \neq \emptyset\} = \min_{\varepsilon_n}\{\varepsilon_n : \bigcap_{s\in\Omega_n} \mathsf{B}(\{s\}, \varepsilon_n) \neq \emptyset\}$ is derived.

From (3.10), the condition to degenerate $\iota(s_i(j))$ to a point is given by $v(s_i(j)) - \varepsilon_n = v(\Omega_n) - v\Big(\Omega_n \setminus \{s_i(j)\}\Big) + \varepsilon_n$ from which we obtain

$$\varepsilon_n^*(i, j) = -\frac{1}{2}\left[v(\Omega_n) - v\Big(\Omega_n \setminus \{s_i(j)\}\Big) - v\Big(\{s_i(j)\}\Big)\right].$$

If ε_n becomes smaller than $\varepsilon_n^*(i, j)$, $\iota(s_i(j))$ that satisfies (3.10) becomes empty. Thus, for $\min_{\varepsilon_n}\{\varepsilon_n : \bigcap_{s\in\Omega_n} \mathsf{B}(\{s\}, \varepsilon_n) \neq \emptyset\}$ to be obtained, it must be set $\varepsilon_n = \max_i \varepsilon_n^*(i, j)$, from which the lemma is derived. □

Continuation of proof of Theorem 4. From Lemma 5, any workers in group $\hat{\imath}$ obtain excess (3.5). Therefore, their payoff $\iota(s_{\hat{\imath}}(j))$ becomes

$$\iota(s_{\hat{\imath}}(j)) = v\Big(\{s_{\hat{\imath}}(j)\}\Big) - \varepsilon_n = \frac{1}{2}v\Big(\{s_{\hat{\imath}}(j)\}\Big) + \frac{1}{2}\left[v(\Omega_n) - v\Big(\Omega_n \setminus \{s_{\hat{\imath}}(j)\}\Big)\right].$$

The $(n+1)$-th reduced game has player set $\Omega_{n+1} = \Omega_n \setminus \bigcap_{j=1}^{N_{\hat{\imath}}}\{s_{\hat{\imath}}(j)\}$, i.e. all players in group $\hat{\imath}$ are removed from the game. Note that player $s_0(1)$ stays in the new game. According to the definition of max-reduced games, its characteristic function becomes

$$v(\Omega_{n+1}) = v(\Omega_n) - \sum_{j=1}^{N_{\hat{\imath}}} \iota(s_{\hat{\imath}}(j)) = v(\Omega) - \bigcup_{s\in\Omega_{n+1}} \iota(s)$$

and, for any $S \subset \Omega_{n+1}$,

$$\begin{aligned} v(S) &= \max\left\{v(S \cup Q) - \sum_{s\in Q} \iota(s) : Q \subseteq \Omega \setminus \Omega_{n+1}\right\} \\ &= \begin{cases} v(S \cup (\Omega \setminus \Omega_{n+1}) - \sum_{s\in\Omega\setminus S} \iota(s) & \text{if } s_0(1) \in S \\ \sum_{s\in S} v(s) & \text{if } s_0(1) \notin S \end{cases} \end{aligned}$$

which confirms that the assumed characteristic function of Ω_n is actually correct by induction.

If $\partial F/\partial l_{ij} \geq U_i$ for all Υ, the game is zero-monotone, and the above lexicographic center is nucleolus (Maschler et al. (1979)). Since core is non-empty from Lemma 2, the nucleolus is included in core. □

Theorem 6. *In $\eth^3_{\infty'}(\Omega, v)$, the following imputation is supported by Shapley value and nucleolus.*

$$E_{ij}(n) = \frac{1}{2}\left(U_i + \frac{\partial F}{\partial l_{ij}}\right)$$

Proof. The result follows from Theorem 1 and Theorem 4. For the latter, it is sufficient to show that F satisfies essential concavity. From concavity of F, $\sum_{i,j}\left(\partial F(\boldsymbol{l}^1)/\partial l_{ij}\right) dl_{ij} \leq \sum_{i,j}\left(\partial F(\boldsymbol{l}^2)/\partial l_{ij}\right) dl_{ij}$. Pick up any type $(\hat{\imath}, \hat{\jmath})$ and set $dl_{ij} = 0$ for any $(i, j) \neq (\hat{\imath}, \hat{\jmath})$. Then, $\left(\partial F(\boldsymbol{l}^1)/\partial l_{\hat{\imath}\hat{\jmath}}\right) dl_{\hat{\imath}\hat{\jmath}} \leq \left(\partial F(\boldsymbol{l}^2)/\partial l_{\hat{\imath}\hat{\jmath}}\right) dl_{\hat{\imath}\hat{\jmath}}$ which implies essential concavity of F, i.e. $F(\boldsymbol{l}^1 + \delta l_{\hat{\imath}\hat{\jmath}}) - F(\boldsymbol{l}^1) \leq F(\boldsymbol{l}^2 + \delta l_{\hat{\imath}\hat{\jmath}}) - F(\boldsymbol{l}^2)$ where $\delta l_{\hat{\imath}\hat{\jmath}}$ denotes the measure of type-$(\hat{\imath}, \hat{\jmath})$ labor. □

We labeled S_0 as a set of a firm or an entrepreneur above. However, if there is any player who exerts non-degenerate influence on productivity or, in other words, those who embodies critical knowledge for production as rent, this player will receive non-marginal part of coalitional rent. In this section, we derived bargaining solution in terms of value function. Its distribution is actually done through wage payment. Bargaining outcome in terms of wages is derived in section 4 and Appendix C.

4. Wage Function

In this section, wage function is derived when there exists some $\tilde{\sigma}_i \geq 0$ for all i such that $\tilde{\sigma}_{ij} = \tilde{\sigma}_i$ for all j. This is the case, for example, if there are multiple declarable types, the natural separation rate is common for all those types and potential demand constraint is unbinding. Also, the condition is obviously satisfied when there exists only one kind of labor in the economy. Since the wage function in general cases is more complicated than presented in this section, it is derived in Appendix C.[13]

By defining $z_{ij} := E_{ij} - U_i$ for all $(i, j) \in \Upsilon$ in Bellman equations (2.4) and (2.5), the dimension of the dynamics is reduced by one:

$$\dot{\boldsymbol{z}}_i(t) = A_i(t)\, \boldsymbol{z}_i(t) - \boldsymbol{\omega}_i(t) \tag{4.1}$$

where $\boldsymbol{z}_i(t) := \begin{pmatrix} z_{i1}(t) \\ z_{i2}(t) \\ \vdots \\ z_{iM_i}(t) \end{pmatrix}$, $A_i(t) := \begin{pmatrix} r(t) + \tilde{\sigma}_i(t) + g_{i1}\mu_i(t) & g_{i2}\mu_i(t) & \cdots & g_{iM_i}\mu_i(t) \\ g_{i1}\mu_i(t) & r(t) + \tilde{\sigma}_i(t) + g_{i2}\mu_i(t) & & g_{iM_i}\mu_i(t) \\ \vdots & & \ddots & \vdots \\ g_{i1}\mu_i(t) & g_{i2}\mu_i(t) & \cdots & r(t) + \tilde{\sigma}_i(t) + g_{iM_i}\mu_i(t) \end{pmatrix}$

and $\boldsymbol{\omega}(t) := \begin{pmatrix} w_{11}(t) - b_1(t) \\ w_{12}(t) - b_1(t) \\ \vdots \\ w_{LM_L}(t) - b_L(t) \end{pmatrix}$. Note that $A(t)$ has eigenvalues $r(t) + \tilde{\sigma}_i(t)$ with multiplicity $(M - 1)$ and $r(t) +$

[13] The arguments below follows the traditional derivation of wages for comparison purpose. However, it would be more straightforward to use "integral version" of Bellman equations as in Appendix C out of steady state.

$\tilde{\sigma}_i(t) + \mu(t)$ with multiplicity one.[14] It can be confirmed that the following provides the elementary matrix $\Phi(t, s)$:

$$\Phi_i(t, s) := e^{\int_s^t A_i(q)dq} = e^{\int_s^t (r(q)+\tilde{\sigma}_i(q))dq} \left[I + \left(e^{\int_s^t \mu_i(q)\, dq} - 1 \right) G \right]$$

where I is an identity matrix and G_i is an "expectation matrix"

$$G_i = \begin{pmatrix} g_{i1} & g_{i2} & \cdots & g_{iM_i} \\ \vdots & \vdots & & \vdots \\ g_{i1} & g_{i2} & \cdots & g_{iM_i} \end{pmatrix}.$$

Namely, $\boldsymbol{z}_i(t) = \Phi_i(t, s)\, \boldsymbol{c}$ for any $\boldsymbol{c} \in \mathbb{R}^2$ solves the accompanying homogeneous equation to (4.1). Then, the solution to (4.1) is given by $\boldsymbol{z}(t) = \Phi(t, 0)[\boldsymbol{z}_0 - \int_0^t \Phi(s, 0)^{-1}\, \boldsymbol{\omega}(s)\, ds] = e^{\int_0^t A(q)dq}[\boldsymbol{z}_0 - \int_0^t e^{-\int_0^s A(q)dq}\, \boldsymbol{\omega}(s)\, ds]$ for any initial value $\boldsymbol{z}(0) = \boldsymbol{z}_0 = (z_{10}, \ldots, z_{L0})$. Note that $\Phi(t, s)^{-1} = e^{-\int_s^t A(q)dq}$. For the no-Ponzi game condition to hold, the initial value must be set at $\boldsymbol{z}_0 = \int_0^\infty \Phi(s, 0)^{-1}\, \boldsymbol{\omega}(s)\, ds$ in which integration is bounded. For such an initial value, $\boldsymbol{z}(t) = \int_t^\infty \Phi(s, t)^{-1}\, \boldsymbol{\omega}(s)\, ds = \int_t^\infty e^{-\int_t^s A(q)dq}\, \boldsymbol{\omega}(s)\, ds$. Using the fact that $[I + (\alpha - 1)\, G]^{-1} = I + \left(\alpha^{-1} - 1\right) G$ for any scholar α, it is found that

$$\Phi_i\, (s, t)^{-1} = e^{-\int_t^s (r+\tilde{\sigma}_i)} \left[I + \left(e^{-\int_t^s \mu_i} - 1 \right) G_i \right].$$

Namely,

$$z_{ij}\,(t) = \int_t^\infty e^{-\int_t^s (r+\tilde{\sigma}_i)} \left[\left(w_{ij} - b_i \right) - \mathsf{E}_j \left(w_{ij} - b_i \right) \right] ds + \int_t^\infty e^{-\int_t^s (r+\tilde{\sigma}_i+\mu_i)} \mathsf{E}_j \left(w_j - b_i \right) ds \tag{4.2}$$

where expectation E is taken over all possible undeclarable worker types. Solving differential equation (2.4) for U_i using (4.2),

$$U_i(t) = \int_t^\infty e^{-\int_t^s r} \left[b_i(s) + \mu_i(s) \int_s^\infty \mathsf{E}_j \left(w_{ij}(\xi) - b_i(\xi) \right) e^{-\int_s^\xi (r+\tilde{\sigma}_i+\mu_i)} d\xi \right] ds \tag{4.3}$$

Similarly, we obtain the value function of employment for each type.

$$\begin{aligned} E_{ij}(t) = & \int_t^\infty e^{-\int_t^s r} w_{ij}(s) ds \\ & + \int_t^\infty ds \int_s^\infty e^{-\int_t^s r} \tilde{\sigma}_i(s) \cdot e^{-\int_s^\xi (r+\tilde{\sigma}_i)} \left[\mathsf{E}_j \left(w_{ij}(\xi) - b_i(\xi) \right) - \left(w_{ij} - b_i \right) \right] d\xi \\ & - \int_t^\infty ds \int_s^\infty e^{-\int_t^s r} \tilde{\sigma}_i(s) \cdot e^{-\int_s^\xi (r+\tilde{\sigma}_i+\mu_i)} \mathsf{E}_j \left(w_{ij}(\xi) - b_i(\xi) \right) d\xi \end{aligned} \tag{4.4}$$

The unemployment value is the discounted series of unemployment benefit and capital gain arising from matching. The employment value is the discounted series of wage rate, expected change of capital gain in new jobs and capital gain (loss) of dismissal.

Proposition 7. *Wage rate at time t is given by*

[14] To obtain this simple result, common separation rate among all undeclarable types of workers is critical.

$$w_{ij}(t) = b_i(t) + \left(\mathsf{E}_h \mathfrak{F}_{ih}(t) - \frac{b_i(t)}{2}\right) + \frac{1}{2}\left(\mathsf{E}_h \mathfrak{F}_{ih}(t) - \mathfrak{F}_{ij}(t)\right) + \left(\tilde{\sigma}_i(t) + \frac{\mu(t)}{2}\right)\int_t^\infty \left(\mathsf{E}_h \mathfrak{F}_{ih}(\xi) - \frac{b_i(\xi)}{2}\right) e^{-\int_t^\xi (r+\mu_i/2)} d\xi + \frac{1}{2}\tilde{\sigma}_i(t)\int_t^\infty \left(\mathsf{E}_h \mathfrak{F}_{ih}(\xi) - \mathfrak{F}_{ij}(\xi)\right) e^{-\int_t^\xi r} d\xi \tag{4.5}$$

where $\mathfrak{F}_i$ is capital gain of marginal value of production, i.e. $\mathfrak{F}_{ij} := r\partial F/\partial l_{ij} - \partial^2 F/\partial t \partial l_{ij}$.

Proof. Theorem 6 implies $\partial^2 F/\partial t\, \partial l_i = 2\dot{E}_i - \dot{U}$. Applying (2.4), (2.5), (4.3) and (4.4),

$$\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} = r(t)\frac{\partial F(t)}{\partial l_{ij}} - \left(2w_{ij}(t) - b_i(t)\right) + 2\tilde{\sigma}_i(t)\, z_{ij}(t) - \mu_i(t)\int_t^\infty \mathsf{E}_h\left(w_{ih} - b_i\right) e^{-\int_t^\xi (r+\mu+\tilde{\sigma}_i)} d\xi \tag{4.6}$$

Taking difference of (4.6) for any i and $j \neq i$, we obtain a Volterra integral equation of the second kind concerning w_{ih} and w_{ij}.[15]

$$\left(w_{ij}(t) - w_{ih}(t)\right) - \tilde{\sigma}_i(t)\int_t^\infty \left(w_{ij}(\xi) - w_{ih}(\xi)\right) e^{-\int_t^\xi (r+\tilde{\sigma}_i)} d\xi = \frac{1}{2}\left[r(t)\left(\frac{\partial F(t)}{\partial l_{ij}} - \frac{\partial F(t)}{\partial l_{ih}}\right) - \left(\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}}\right)\right] \tag{4.7}$$

On the other hand, taking expectation of (4.6) yields

$$\mathsf{E}_h\left(w_{ih}(t) - b_i(t)\right) - \left(\tilde{\sigma}_i(t) - \frac{\mu_i(t)}{2}\right)\int_t^\infty \mathsf{E}_{ih}\left(w_{ih}(\xi) - b_i(\xi)\right) e^{-\int_t^\xi (r+\tilde{\sigma}_i+\mu_i)} d\xi = \frac{1}{2}\mathsf{E}_h\left(r(t)\frac{\partial F(t)}{\partial l_{ih}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}} - b_i(t)\right) \tag{4.8}$$

The above results suggest that it is beneficial to define new variables $Y_{ij}(t)$ ($j = 1, 2, \ldots, M_i$) as follows.

$$\begin{pmatrix} Y_{i1}(t) \\ Y_{i2}(t) \\ Y_{i3}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} := \left(\begin{array}{c|cccc} g_{i1} & g_{i2} & g_{i3} & \cdots & g_{iM_i} \\ \hline 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{array}\right) \begin{pmatrix} w_{i1}(t) - b_i(t) \\ w_{i2}(t) - b_i(t) \\ w_{i3}(t) - b_i(t) \\ \vdots \\ w_{iM_i}(t) - b_i(t) \end{pmatrix} \tag{4.9}$$

Observe that the above conversion matrix is the same as the eigenvector matrix of $A_i(t)$. By this change of variables, we can "diagonalize" the simultaneous integral equations concerning w_{ij}'s, (4.7) and (4.8). Namely,

$$\begin{pmatrix} Y_{i1}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} - \int_t^\infty \begin{pmatrix} K_{i1}(t,\xi) & & O \\ & \ddots & \\ O & & K_{iM_i}(t,\xi) \end{pmatrix} \begin{pmatrix} Y_{i1}(\xi) \\ \vdots \\ Y_{iM_i}(\xi) \end{pmatrix} d\xi = \frac{1}{2}\begin{pmatrix} h_{i1} \\ \vdots \\ h_{iM} \end{pmatrix}$$

where

$$K_{i1}(t,\xi) := \left(\tilde{\sigma}_i(t) - \frac{\mu_i(t)}{2}\right) e^{-\int_t^\xi (r+\sigma_i+\mu_i)}$$

$$K_{ij}(t,\xi) := \tilde{\sigma}_i(t)\, e^{-\int_t^\xi (r+\tilde{\sigma}_i)} \qquad (\text{for all } j = 2, \ldots, M_i)$$

[15]Note that it is impossible to obtain a differential equation by taking time derivative of this equation since t resides inside of the integration. It is a general consequence of non-stationarity.

$$h_{i1}(t) := \mathsf{E}_h\left(r(t)\frac{\partial F(t)}{\partial l_{ih}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}} - b_i(t)\right)$$
$$h_{ij}(t) := r(t)\left(\frac{\partial F(t)}{\partial l_{ij}} - \frac{\partial F(t)}{\partial l_{1j}}\right) - \left(\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} - \frac{\partial^2 F(t)}{\partial t \partial l_{1j}}\right) \qquad \text{(for all } j = 2, \ldots, M)$$

and the integration is applied element-wise. Then, the solution to this equation is given by

$$\begin{pmatrix} Y_{i1}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} h_{i1}(t) \\ \vdots \\ h_{iM_i}(t) \end{pmatrix} - \frac{1}{2}\int_t^\infty \begin{pmatrix} G_{i1}(t,\xi) & & O \\ & \ddots & \\ O & & G_{iM_i}(t,\xi) \end{pmatrix}\begin{pmatrix} h_{i1}(\xi) \\ \vdots \\ h_{iM_i}(\xi) \end{pmatrix} d\xi$$

where $G_{ij}(t,\xi) := -\sum_{\zeta=1}^{\infty} \overset{*}{K}{}_{ij}^{\zeta}(t,\xi)$ for $j = 1, 2, \ldots, M_i$. Iterated kernel $\overset{*}{K}{}^n$ is defined by $\overset{*}{K}{}^n := \underbrace{K * K * \cdots * K}_{n}$ and $K * L$ denotes the composition of the first kind defined by $K(t,\xi) * L(t,\xi) = \int_t^\xi K(t,\tau)\, L(\tau,\xi)\, d\tau$ (see e.g. Yokota (2006) and other literature on integral equations). Since

$$\overset{*}{K}{}_{i1}^{n}(t,\xi) = \left(\tilde{\sigma}_i(t) + \frac{\mu_i}{2}(t)\right) e^{-\int_t^\xi (r+\sigma_i+\mu_i)} \frac{\left[\int_t^\xi \left(\tilde{\sigma}_i(s) + \frac{\mu_i(s)}{2}\right) ds\right]^{n-1}}{(n-1)!}$$
$$\overset{*}{K}{}_{ij}^{n}(t,\xi) = \tilde{\sigma}_i(t) e^{-\int_t^\xi (r+\sigma_i)} \frac{\left[\int_t^\xi \tilde{\sigma}_i(s)\, ds\right]^{n-1}}{(n-1)!} \qquad \text{(for all } j = 2, \ldots, M),$$

we obtain

$$G_{11}^i(t,\xi) = -\left(\tilde{\sigma}_i(t) + \frac{\mu_i(t)}{2}\right) e^{-\int_t^\xi (r+\mu_i/2)}$$
$$G_{jj}^i(t,\xi) = -\tilde{\sigma}_i(t)\, e^{-\int_t^\xi r} \qquad \text{(for all } j = 2, \ldots, M)$$

and the solution for $Y_{ij}(t)$:

$$Y_{i1}(t) = \frac{1}{2} h_{i1}(t) + \frac{1}{2}\left(\tilde{\sigma}_i(t) + \frac{\mu_i(t)}{2}\right) \int_t^\infty e^{-\int_t^\xi (r+\mu_i/2)} h_{i1}(\xi)\, d\xi$$
$$Y_{ij}(t) = \frac{1}{2} h_{ij}(t) + \frac{1}{2}\tilde{\sigma}_i(t) \int_t^\infty e^{-\int_t^\xi r} h_{ij}(\xi)\, d\xi \qquad \text{(for all } j = 2, \ldots, M).$$

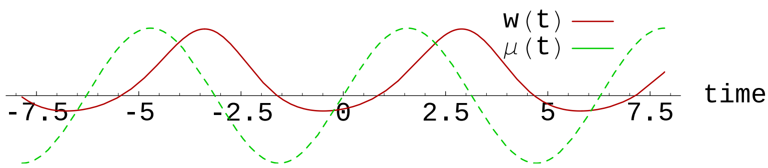
Inverting back to $w_{ij}(t)$ using the inverse function of (4.9), i.e.

$$\begin{pmatrix} w_{i1}(t) - b_i(t) \\ w_{i2}(t) - b_i(t) \\ \vdots \\ w_{iM}(t) - b_i(t) \end{pmatrix} = \left[\begin{pmatrix} 1 & g_{i2} & \cdots & g_{iM_i} \\ 1 & g_{i2} & \cdots & g_{iM_i} \\ \vdots & \vdots & & \vdots \\ 1 & g_{i2} & \cdots & g_{iM_i} \end{pmatrix} - \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & & O \\ \vdots & & \ddots & \\ 0 & O & & 1 \end{array}\right)\right]\begin{pmatrix} Y_{i1}(t) \\ Y_{i2}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix},$$

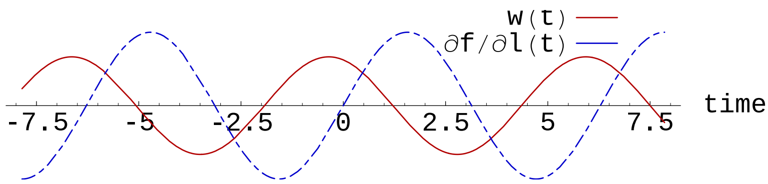
the result of the proposition is derived. □

Note that, whereas wage rate responds immediately to the change of separation rate, *its response to the change of matching rate or marginal production value accompanies time-lag*. It is caused by the fact that, while the

(1) Response of wages w to matching rate μ



(2) Response of wages w to marginal value of production $\partial f/\partial l$



Both show reaction of w to the forced oscillation in matching rate and in marginal productivity, respectively, where $\mu(t) = \partial f/\partial l = 1 + \sin t$ is assumed. Vertical constants are arbitrarily adjusted so that the phase shift is easily visible. Used parameters are: $b = 1$, $\tilde{\sigma} = 1$, $r = 0.05$, $\mathcal{F} = 1$ for the first graph and $b = 1$, $\mu = 1$, $r = 0.05$, $t^e = 20$ for the second where t^e is the entering time to the bounded surface.

Figure 4.1: Response of wage rate to forced oscillation

adjustment of workforce through dismissal is achieved promptly, the matching process is time-consuming. To see the latter fact, suppose that there is only one kind of labor. Matching rate μ fluctuates according to $\mu(t) = 1 + \sin t$ and other unrelated variables are fixed. Then, from equation (4.5), wage rate is given by

$$\begin{aligned} w(t) &= C_0 + C_1\left(C_2 + \frac{\mu(t)}{2}\right)\int_t^{\infty} \exp\left[-\int_t^{\xi}\left(C_3 + \frac{\mu(s)}{2}\right)ds\right]d\xi \\ &= C_0 + C_1\left(C_2 + \sin t\right)e^{-(\cos t - C_3 t)/2}\int_t^{\infty} e^{(\cos\xi - C_3\xi)/2}d\xi \end{aligned}$$

where C_i $(i = 1, 2, 3)$ are indeterminate constants. The functional form is drawn in the first graph of Figure 4.1. The second graph shows the response of wages against the change in marginal productivity. Note that marginal productivity is a decreasing function of y. It will be shown that, when a firm is operating below the demand surface, the marginal production value is given by $\partial F/\partial l = \int_t^{t^e} \partial f/\partial l\, e^{-\int_t^{\xi}(r+\tilde{\sigma})}d\xi$ where t^e is the entering time to the demand surface. It assumes $\partial f/\partial l = 1 + \sin t$,[16] which implies $\mathfrak{F} = p\sin t - q\cos$ for some p and q, therefore the wage function becomes

$$\begin{aligned} w(t) &= C_0 + \mathfrak{F} + C_1\int_t^{\infty}\left(C_2 + \mathfrak{F}\right)e^{-C_3(\xi - t)}d\xi \\ &= C_0 + \frac{r\sin t - \cos t}{2} + C_1 e^{C_3 t}\int_t^{\infty}\left(C_2 + \frac{r\sin\xi - \cos\xi}{2}\right)e^{-C_3\xi}d\xi \end{aligned}$$

These effects of matching rate and marginal productivity would show more or less synchronized behavior in the actual economy, since they are countercyclical from each other. The lagged response shown above is not limited

[16] The case of binding demand constraint can be derived in a similar way where the only difference is that a cyclical component is introduced in the discount factor. On the other hand, if the economy is about to leave the demand constraint, $\partial F/\partial l$ is not a simple integration of marginal productivity.

to a special case where intertemporal fluctuation of μ or $\partial f/\partial l$ is represented by a sine curve. As far as they are absolutely integrable in terms of time, similar properties would be shown via Fourier transformation.

The above effect distorts the share between entrepreneurs and workers over business cycles. It can have real effects when the aggregate demand is a function of the relative distribution between these two groups. Such examples include the case where entrepreneurs have different saving ratio from workers, and the case where the investment decision of firms is a positive function of profits.

Corollary 8. *At steady state with one kind of labor, the wage rate w satisfies the following relation*

$$\frac{\partial F}{\partial l} = \left(1 - \frac{\tilde{\sigma}}{r+\mu+\tilde{\sigma}}\right)\frac{w}{r} + \frac{\tilde{\sigma}}{r+\mu+\tilde{\sigma}}\frac{b}{r} + \frac{r}{r+\mu+\tilde{\sigma}}\left(\frac{w}{r} - \frac{b}{r}\right). \tag{4.10}$$

Furthermore, if $w > b$, then

$$\frac{\partial F}{\partial l} > \frac{w}{r+\tilde{\sigma}}$$

holds.

Proof. Equation (4.10) is obtained by setting $L = 1$, $M_1 = 1$ and all related variables to be at steady state. For the latter half, we see that $\partial F/\partial l > w/(r+\tilde{\sigma})$ is equivalent to $(r^2 + \tilde{\sigma}r + \mu\tilde{\sigma})w > (r^2 - \tilde{\sigma}^2)b$ using equation (4.10). If $w > b$, then $(r^2 + \tilde{\sigma}r + \mu\tilde{\sigma})w > r^2 w > r^2 b > (r^2 - \tilde{\sigma}^2)b$, which shows that this proposition is true. □

The steady state mentioned in the corollary can be either unbounded or bounded steady states. It shows that, as far as work is more preferable than staying unemployed for workers, *the firm is willing to employ more workers once there arises additional demand for output*. Furthermore, it should be observed that if output increases, the marginal productivity of labor, i.e. the left-hand side in equation (4.10), decreases. Thus, *increase of output is achieved through the decrease of real wage rate in the wage bargaining, ceteris paribus*. This result largely depends on our setup assumptions that labor intensity is constant, that workers are not allowed to do overtime work and that profits are not redistributed to workers, say, to provide incentives for efforts.

Now, we can present some general properties on wages. First, the expected present value of wages is generally greater than that of unemployment benefits as far as there remains production opportunities. Second, if the marginal contribution to the value of production is decreasing over time, wage rate is greater than unemployment benefit.

Proposition 9. *If $\partial F/\partial l_{ij} > U_i$ for all (i, j), then $\int_t^\infty \mathsf{E}_j w_{ij}(s) e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds > \int_t^\infty b_i(s)\, e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds$.*

Proof. From Theorem 1, the condition $\partial F/\partial l_{ij} > U_i$ implies $z_{ij} = (\partial F/\partial l_{ij} - U_i)/2 > 0$. Namely,

$$z_{ij} = \int_t^\infty \mathsf{E}_h\left[w_{ih} - b_i\right] e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds - \int_t^\infty \left\{\mathsf{E}_h\left[w_{ih} - b_i\right] - \left(w_{ij} - b_i\right)\right\} e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds > 0$$

must hold for all (i, j) from (4.2), which yields

$$\int_t^\infty \mathsf{E}_h\left[w_{ih} - b_i\right] e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds > \max_j \int_t^\infty \left\{\mathsf{E}_h\left[w_{ih} - b_i\right] - \left(w_{ij} - b_i\right)\right\} e^{-\int(r+\tilde{\sigma}_i+\mu_i)} ds \ \geq\ 0$$

to obtain the result. □

Proposition 10. *If $\partial^2 F/\partial t\partial l_{ij} - \dot{U}_i \leq 0$, then $w_{ij}(t) > b_i(t)$ for all $(i, j) \in \Upsilon$ and t.*

Proof. From Theorem 6,

$$\dot{E}_{ij}(t) = \frac{1}{2}\left(\dot{U}_i(t) + \frac{\partial^2 F}{\partial t\partial l_{ij}}(t)\right) \tag{4.11}$$

which yields

$$\dot{E}_{ij}(t) - \dot{U}_i(t) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial t\partial l_{ij}}(t) - \dot{U}_i(t)\right) \leq 0$$

From (2.4) and (2.5)

$$w_{ij} - b_i = (r + \tilde{\sigma}_i)\left(E_{ij} - U_i\right) + \mu \mathsf{E}_h\left[E_{ih} - U_i\right] - \left(\dot{E}_{ij} - \dot{U}_i\right) > 0.$$

□

The condition of Proposition 10 obviously holds at a steady state either when the demand constraint is binding or unbinding. On the other hand, when b is expected to rise only for a sufficiently short period of time from now on, it can happen that wage rate becomes temporarily smaller than unemployment benefit whereas $E_i > U$ still holds and thus workers do not willing to quit the current jobs.

5. Production Plan

The results of the previous section show that the wage rate is a function of employment. Based on rational expectation on wage schedule $w(\boldsymbol{l})$, the firm determines optimal policy on vacancy post and dismissal. The optimal problem for the firm is given by

$$J(\boldsymbol{l}, y) = \max_{m,x} \int_t^\infty \left[f(\boldsymbol{l}) - \boldsymbol{w}(\boldsymbol{l}) \cdot \boldsymbol{l} - \sum_{i=1}^{L} \kappa_i(m_i)\right] \exp\left[-\int_t^\xi r(\tau) d\tau\right] d\xi \tag{P}$$

subject to

$$\dot{l}_{ij} = g_{ij}\psi(\theta_i)\, m_i - \sigma_{ij} l_{ij} - x_{ij}, \quad \forall i = 1, \ldots, L;\ j = 1, \ldots, M_i \tag{2.1}$$

$$0 \leq x_{ij} \leq X \qquad \forall i, j \tag{5.1}$$

$$m_i \geq 0 \qquad \forall i \tag{5.2}$$

$$f(\boldsymbol{l}) \leq y \tag{5.3}$$

$$l_{ij} \geq 0 \qquad \forall i, j \tag{5.4}$$

$$l_{ij}(0), \forall i, j \text{ given.}$$

where parameters y, g, θ, σ are generally time-dependent and X is an arbitrarily large number. X is assumed to be large enough so that a firm can accommodate any negative change of y. r is bounded and $r(t) \nrightarrow 0$ as $t \to \infty$.

Also, $y(t) \in C^2$ as of the time of planning. It can show *ex post* indifferentiability as a result of unexpected shift of y. It will be proven that labor market constantly shows the state of long-run excess demand below unbounded steady states. Namely, if the firm can employ additional workers due to the increase of y, then it can increase profits. Walras Law implies that the goods market is always in the state of excess supply *regardless the relative price between output goods and labor*. On the other hand, the presence of a convex vacancy cost function prohibits discrete increase of employment, which implies that aggregate production and income can grow only continuously from the current level, and thus the excess supply in the goods market will not be resolved.

Denote the costate variables corresponding to each transition equation of l_{ij} by λ_{ij}. An augmented Hamiltonian H is defined by

$$H(\xi) := f(\boldsymbol{l}) - \mathbf{w}(\boldsymbol{l}) \cdot \boldsymbol{l} - \sum_{i=1}^{L} \kappa_i(m_i) + \sum_{i,j} \lambda_{ij}\left(\phi_{ij} m_i - \sigma_{ij} l_{ij} - x_{ij}\right) + \mu_0 \left(\dot{y} - \sum_{i=ij} \frac{\partial f}{\partial l_{ij}} \dot{l}_{ij}\right) + \sum_{i,j} \mu_{ij}^1 x_{ij} + \sum_{i,j} \mu_{ij}^2 \left(X - x_{ij}\right) + \sum_i \gamma_i m_i \tag{5.5}$$

where $R(t,\xi) := \int_t^\xi r(\tau)\, d\tau$ and μ_0, $\mu_{ij}^n \geq 0$ for $\forall i, j, n$ and $\gamma_i \geq 0$ for $\forall i$ are Lagrange multipliers such that any terms including them are zero. From maximization of Hamiltonian function, optimal conditions for m_i are given by

$$\kappa_i'(m_i) = \sum_j \phi_{ij}\left(\lambda_{ij} - \mu_0 f_{ij}\right) + \gamma_i \tag{5.6}$$

$$\gamma_i m_i = 0 \tag{5.7}$$

$$\lambda_{ij} - \mu_0 f_{ij} = \mu_{ij}^1 - \mu_{ij}^2 \tag{5.8}$$

where $f_{ij} := \partial f / \partial l_{ij}$, and costate dynamics is given by

$$\dot{\lambda}_{ij} = \left(r + \sigma_{ij}\right)\lambda_{ij} + \mu_0\left(\dot{f}_{ij} - \sigma_{ij} f_{ij}\right) - \left(f_{ij} - c_{ij}\right) \qquad \forall i, j \tag{5.9}$$

where $\dot{f}_{ij} := \sum_{a,b} (\partial^2 f / \partial l_{ij} \partial l_{ab})\, \dot{l}_{ab}$, $\mu_0 > 0$ when the demand constraint is binding and $\mu_0 = 0$ when not.

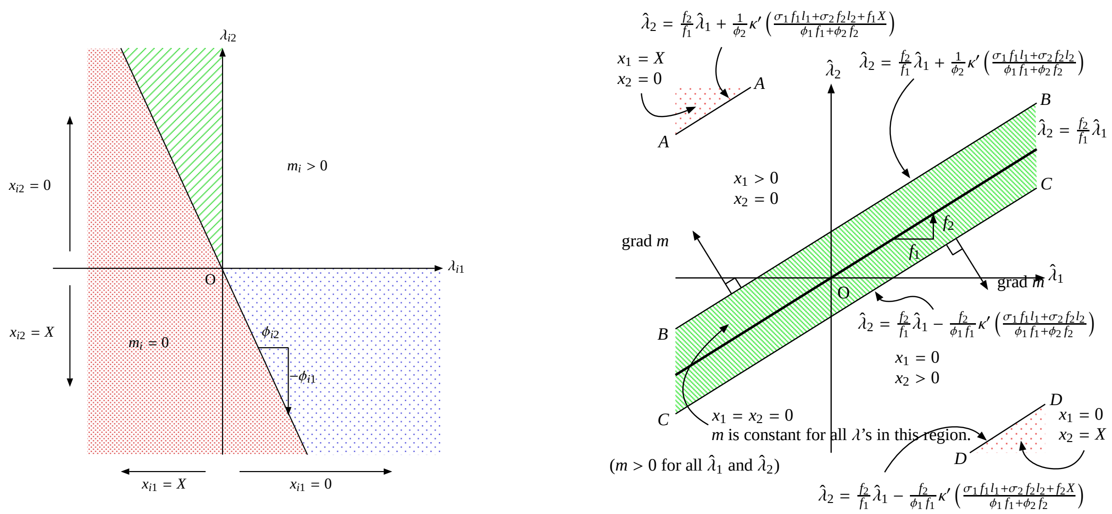
5.1. **Optimal control.**

(a) Off demand constraints. If the demand condition (5.3) is not binding, we have $\mu_0 = 0$. Then, the optimal condition for x is given by

$$x_{ij} = \begin{cases} 0 & \text{if } \lambda_{ij} > 0 \\ X & \text{if } \lambda_{ij} < 0 \end{cases} \qquad \forall i, j \tag{5.10}$$

Proposition 11. *When $f(\boldsymbol{l}) < y$, if $\sum_j \phi_{ij}\lambda_{ij} > 0$, then $m_i > 0$. If $\sum_j \phi_{ij}\lambda_{ij} \leq 0$, then $m_i = 0$.*

Proof. If $\sum_j \phi_i \lambda_i > 0$, the right-hand side of equation (5.6) is strictly positive, which implies $m_i > 0$. If $\sum_j \phi_{ij}\lambda_{ij} <$ 0, then $\gamma_i > 0$ since the left-hand side of equation (5.6) must be non-negative. From equation (5.7), it implies



(a) Off demand constraints (b) On demand constraints

Figure 5.1: Optimal control

$m_i = 0$. If $\sum_j \phi_{ij}\lambda_{ij} = 0$, then equation (5.6) becomes $\kappa'(m_i)\, e^{-R(t,\xi)} = \gamma_i$. If we assume $\gamma_i > 0$, equation (5.6) implies $m_i > 0$, contradicting equation (5.7). Thus, $\gamma_i = m_i = 0$. □

Corollary 12. *If $f(\boldsymbol{l}) < y$ and $m_i > 0$, then $x_{ij} = 0$ for all j.*

Figure 5.1 shows the case of $L = 1$ and $M_1 = 2$.

(b) On demand constraints. When the demand constraint (5.3) is binding, it imposes restrictions on controls in the form of $\sum_{i,j} f_{ij}\dot{l}_{ij} = \dot{y}$, or

$$\sum_i \left(\sum_j \phi_{ij} f_{ij}\right) m_i = \dot{y} + \sum_{i,j} \left(\sigma_{ij} l_{ij} + x_{ij}\right). \tag{5.11}$$

Since (5.11) is constraint expressed in differential form, the initial condition must be provided at the conjunction time. However, given that the path is on the constraint surface in the neighborhood of the present time, (5.11) suffices.

Proposition 13. *Define $k^i_{ab}(\lambda_a, \lambda_b; \boldsymbol{l}) := \lambda_{ia}/f_{ia} - \lambda_{ib}/f_{ib}$.*

(1) *If $\sum_a \phi_{ia} f_{ia} k^i_{aj} \le \kappa'_i(\bar{m}_i)$ for all i and j where $\bar{m}_i$ is a solution to*

$$\dot{y} = \sum_i \left(\sum_j \phi_{ij} f_{ij}\right) \bar{m}_i - \sum_i \sum_j \sigma_{ij} f_{ij} l_{ij}$$
$$\frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa'_i(\bar{m}_i)}{\sum_a \phi_{ia} f_{ia}} = \frac{\sum_a \phi_{i'a}\lambda_{i'a} - \kappa'_{i'}(\bar{m}_{i'})}{\sum_a \phi_{i'a} f_{i'a}}, \quad \forall i, i',$$

then $m^*_i = \bar{m}_i$ and $x_{ij} = 0$ for all i and j.

(2) *If set* $S := \{(i, j) : \sum_a \phi_{ia} f_{ia} k^i_{aj} > \kappa'_i(\bar{m}_i)\}$ *is non-empty, then* m_i *is determined by*

$$\kappa'_{i'}(m_{i'}) = \sum_a \phi_{i'a} f_{i'a} \left(\frac{\lambda_{i'a}}{f_{i'a}} - \frac{\lambda_{i'j}}{f_{i'j}} \right) > \kappa'_{i'}(\bar{m}_{i'}) \qquad \forall i' \in S$$

$$\frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa'_i(\bar{m}_i)}{\sum_a \phi_{ia} f_{ia}} = \frac{\sum_a \phi_{i'a}\lambda_{i'a} - \kappa'_{i'}(\bar{m}_{i'})}{\sum_a \phi_{i'a} f_{i'a}}, \qquad \forall i \notin S.$$

On the other hand, $x_{ij} = 0$ *for all* $(i, j) \notin S$ *and* $x_{i'j'}$ *for all* $(i', j') \in S$ *is given by*

$$\sum_{(i',j')\in S} f_{i'j'} x_{i'j'} = \sum_i \left(\sum_j \phi_{ij} f_{ij} \right) \bar{m}_i - \sum_i \sum_j \sigma_{ij} f_{ij} l_{ij} - \dot{y}$$

and distribution among $x_{i'j'}$*'s is indeterminate.*

Proof. Define $A_{ij} := \lambda_{ij} - \mu_0 f_{ij}$. From (5.8),

$$x_{ij} = \begin{cases} 0 & \text{if } A_{ij} > 0 \\ [0, X] & \text{if } A_{ij} = 0 \\ X & \text{if } A_{ij} < 0 \end{cases}$$

Since X is sufficiently large, the condition $x \leq X$ is never binding, which implies that $A_{ij} \geq 0$ for all i, j and thus $\sum_a \phi_{ia} A_{ia} \geq 0$ for all i. Then, since $\gamma_i = 0$ for all i, $\kappa'_i(m_i) = \sum_a \phi_{ia} A_{ia}$. Solving this obtains

$$\mu_0 = \frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa'_i(m_i)}{\sum_a \phi_{ia} f_{ia}} \tag{5.12}$$

for all i. First, suppose $x_{ij} = 0$ for all i and j. (5.11) and (5.12) together with $x_{ij} = 0$ determines m_i which is common for all range of $x_{ij} = 0$ for all i, j for given $\boldsymbol{l}$. Let us denote it by $\bar{m}_i$. Then, from $A_{ij} \geq 0$, the condition $x_{ij} = 0$ for all i, j is equivalent to domain $\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa'_i(\bar{m}_i)$ for all i and j. Next, suppose that there exist some i' and j' such that $x_{i'j'} > 0$. Then, from $\mu_0 = \lambda_{i'j'} / f_{i'j'}$,

$$\kappa'_{i'}(m_{i'}) = \sum_a \phi_{i'a} f_{i'a} \left(\frac{\lambda_{i'a}}{f_{i'a}} - \frac{\lambda_{i'j'}}{f_{i'j'}} \right)$$

From the demand constraint (5.11), $\sum_i (\sum_a \phi_{ia} f_{ia})(m_i - \bar{m}_i) > 0$. On the other hand, from (5.12), if $m_i \gtreqless \bar{m}_i$ for some i, then $m_j \gtreqless \bar{m}_j$ for any j. These leads to $m_i > \bar{m}_i$ for all i. □

Optimal control for each λ for the case of two undeclarable types is shown in Figure 5.1 (b). $|k^i_{ab}|$ can be interpreted as pressure that represents the necessity for structural change in employment composition between type a and b. If the pressure is sufficiently weak, the structural change is achieved solely through the adjustment of new employment and natural separation. As the pressure grows, the firm is compelled to adopt dismissal. The bandwidth of the no-dismissal domain that gives $x_i = 0$ for $\forall i$ positively depends on l's. Namely, larger firms in employment size less-likely adopt dismissal and rejection for structural adjustment of labor for given λ, i.e. for given influence of labor on profit value. The linear contour structure of optimal control on the demand constraint shown in Figure 5.1 (b) is a direct consequence of the presence of the demand constraint. It is related to the fact that the demand constraint defines the first integral which brings Hamiltonian-invariant vector field along the contours. Namely, it reflects the symmetricity between such a vector field and the original Hamiltonian field that H generates, in the sense that their Hamiltonians are Poisson commuting. To observe it, we need the following theorem.

Theorem 14 (Noether's Theorem)**.** *For function H defined on a simply-connected domain, the following two statements are equivalent.*

(1) *There exists a function G such that*
   (a) *G is not a constant function and*
   (b) *G satisfies $\{G, H\} = 0$ where $\{\cdot,\cdot\}$ is Poisson brackets.*
(2) *There exists a one-parameter group of transformation with parameter s, φ_s, such that*
   (a) *φ_s is a canonical transformation,*
   (b) *φ_s is not an identity transformation.*

   Moreover, the vector field brought by infinitesimal transformation of φ_s corresponds the Hamiltonian vector field where Hamiltonian is given by G.

Obviously, the demand constraint $G(\boldsymbol{l}, \boldsymbol{\lambda}) = f(\boldsymbol{l}) - y = 0$ has constant value over time, so it becomes one of the first integrals which is not a constant function in terms of $(\boldsymbol{l}, \boldsymbol{\lambda})$ and which satisfies $\{G, H\} = \sum_{i,j} \partial G/\partial l_{ij}\, \partial H/\partial \lambda_{ij} = 0$.[17] Noether's Theorem tells us that the vector field which is brought by the infinitesimal transformation of φ_s should possess G as its Hamiltonian, i.e. the following relationship should hold:

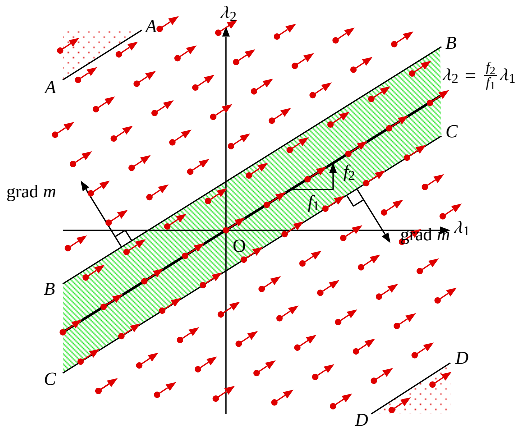
$$\left.\frac{d\varphi_s}{ds}\right|_{s=0} = \sum_{i,j} f_{ij} \frac{\partial}{\partial \lambda_{ij}}. \tag{5.13}$$

Note that, since $\partial G/\partial \lambda_{ij} = 0$ for any i, j, $\partial/\partial l_{ij}$ terms vanish. This vector field is shown in Figure 5.2 for the case of two kinds of labor by identifying the differential operator with the basis of the tangent space. How can we construct φ_s that satisfies conditions (2) in Theorem 14 and therefore equation (5.13)? By integrating (5.13), we can think of the following class of action of group $\mathbb{R}$ (denoted by $s \in \mathbb{R}$) on $\mathbb{R}^{2\sum_i M_i}$, denoted by φ_s, as a candidate:

$$\begin{pmatrix} L_{11} \\ \vdots \\ L_{LM_L} \\ \hline \Lambda_{11} \\ \vdots \\ \Lambda_{LM_L} \end{pmatrix} = C(\boldsymbol{l}, \boldsymbol{\lambda}) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline f_{11} \\ \vdots \\ f_{LM_L} \end{pmatrix} s =: \varphi_s \tag{5.14}$$

where $(L_{11}, \ldots, L_{LM_L})$ is the state vactor and $(\Lambda_{11}, \ldots, \Lambda_{LM_L})$ is the costate vector after transformation, respectively, and $*$ is an arbitrary function. This transformation leaves the optimal control invariant, thus it can be a

[17]The property that the first integral G of Hamiltonian dynamics characterized by Hamiltonian H satisfies $\{G, H\} = 0$ is a general result and its inverse also holds.



The Hamiltonian of this vector field becomes the first integral in the Hamiltonian field in the configuration space.

Figure 5.2: Vector field in the conjugate space brought by the demand constraint

candidate to preserve Hamiltonian that condition (2a) of Theorem 14 designates. Since $f_{ij} \neq 0$, φ_s is not an identity transformation, i.e. condition (2b) holds. The remaining task is to find out appropriate s and $C(\boldsymbol{l}, \boldsymbol{\lambda})$ in equation (5.14) to make φ_s a canonical transformation to satisfy condition (2a).

Choose $(i^*, j^*) \in \Upsilon$ arbitrarily and set $s = -\lambda_{i^* j^*}/(1 + f_{i^* j^*})$ and

$$C(\boldsymbol{l}, \boldsymbol{\lambda}) = {}^t\Big(l_{11} \quad \cdots \quad l_{i^*, j^*-1} \quad l_{i^* j^*} + f(\boldsymbol{l}) - y \quad l_{i^*, j^*+1} \quad \cdots \quad l_{LM_L} \;\Big|\; \lambda_{11} \quad \cdots \quad \lambda_{LM_L} \Big)$$

accordingly to satisfy equation (5.14). This makes this change of variables a canonical transformation. We assume $i^* = j^* = 1$ without loss of generality. Denote the transformation by $\Phi : (\boldsymbol{L}; \boldsymbol{\Lambda}) \to (\boldsymbol{l}; \boldsymbol{\lambda}) = (\varphi(\boldsymbol{L}); \boldsymbol{\lambda})$ such that

$$\varphi^{-1} : \begin{pmatrix} L_{11} \\ L_{12} \\ \vdots \\ L_{LM_L} \end{pmatrix} = \begin{pmatrix} l_{11} + y(t) - f(\boldsymbol{l}) \\ l_{12} \\ \vdots \\ l_{LM_L} \end{pmatrix} \qquad \text{or equivalently} \qquad \varphi : \begin{pmatrix} l_{11} \\ l_{12} \\ \vdots \\ l_{LM_L} \end{pmatrix} = \begin{pmatrix} g(t, \boldsymbol{L}) \\ L_{12} \\ \vdots \\ L_{LM_L} \end{pmatrix}.$$

Then, since Φ defined above satisfies

$$\begin{pmatrix} \Lambda_{i1} \\ \vdots \\ \Lambda_{iM_i} \end{pmatrix} = {}^t\varphi_L \begin{pmatrix} \lambda_{i1} \\ \vdots \\ \lambda_{iM_i} \end{pmatrix},$$

this is a variant of point transformation, a special case of canonical transformation.[18]

In sum, the linear contour map shown in Figure 5.2 is a reflection that, the vector field corresponding to optimal-control-preserving —and thus Hamiltonian-preserving— φ_s has the demand constraint as Hamiltonian.

[18]See books on analytical mechanics, e.g. Arnol'd (1989) and Ito (1998).

5.2. **Consideration on possible discontinuities of control variables on junction points.** If the interest rate fully adjusts intertemporal demand, the economy is on the demand surface at any time in the market equilibrium. However, if any disturbances are added on interest rates for some reason—such as monetary shocks—, then it can be away from the constraint. The principle of dynamic programming tells us that there can be time-discontinuity in costate variables and thus in control variables. However, this section shows that it never happens with the model in consideration.

Entering condition to the demand constraint. Let $C \subset \mathbb{R}^M$ be a configuration space. Define an *entering time* $t^e \in \mathbb{R}$ to a state constraint surface $\mathcal{B} \subset C$ such that the Lagrange variable μ_0 adjoint to the state constraint $\mathcal{B}$ yields $\mu_0(t^e) = \mu_0(t^e - \varepsilon) = 0$ and $\mu_0(t^e + \varepsilon) > 0$ for any arbitrarily small $\varepsilon > 0$. Let $z(t) : \mathbb{R} \to C$ be a path, i.e. a trajectory projected onto the configuration space. Define $z(t^e)$ as an *entering point*. Similarly, *leaving time* $t^l \in \mathbb{R}$ from a state constraint $\mathcal{B}$ is defined to be $\mu_0(t^l) = \mu_0(t^l + \varepsilon) = 0$ and $\mu_0(t^l - \varepsilon) > 0$ for any arbitrarily small $\varepsilon > 0$. $z(t^l)$ is called a *leaving point*. Denote a set of all entering time by T^e and a set of all leaving time by T^l. We also call $t^j \in T^e \cap T^l$ a junction time and $z(t^j)$ a junction point. In general, costate variables can show time-discontinuity either at entering or leaving points (see e.g. Bryson et al. (1963)). This is due to the fact that, over time, the state constraint separates the normal of intertemporal transformation of the neighborhood of the optimal trajectory on the limiting surface at entering or leaving time from the normal of the limiting surface itself. Despite the fact, for the current problem, it turns out that costate variables are actually continuous both at entering and leaving time. This is due to the one-way property of the path, i.e. as far as no external force is added on y, the path permanently stays on the demand surface. At conjunction time $t \in T^e \cup T^l$,

$$\lambda_{ia}^- = \lambda_{ia}^+ + \rho f_{ia} \tag{5.15}$$

$$H^- = H^+ + \rho\dot{y} \tag{5.16}$$

must hold where ρ is a Lagrange variable adjoint to the state constraint $f(\boldsymbol{l}) - y = 0$ and, for any variable A, we denote $A^- := \lim_{t\uparrow t^J} A$, $A^+ := \lim_{t\downarrow t^J} A$. From (5.15),

$$\rho = \frac{\lambda_{ia}^- - \lambda_{ia}^+}{f_{ia}}$$

for all i and a, which implies

$$\Delta k_{ab}^i = \frac{\Delta\lambda_{ia}}{f_{ia}} - \frac{\Delta\lambda_{ib}}{f_{ib}} = 0 \tag{5.17}$$

for all i, a and b such that $a \neq b$. It implies that costate variables jump at entering points along contour lines of m as shown in Figure 5.1 so that k_{ab}^i does not change. From (5.16),

$$\sum_{i,j} \lambda_{ij}^- \left(\phi_{ij} m_i^- - \sigma_{ij} l_{ij} - x_{ij}^-\right) - \sum_i \kappa_i \left(m_i^-\right) = \sum_{i,j} \lambda_{ij}^+ \left(\phi_{ij} m_i^+ - \sigma_{ij} l_{ij} - x_{ij}^+\right) - \sum_i \kappa_i \left(m_i^+\right) + \rho\dot{y}$$

Using (5.15), it turns out

$$\sum_i \kappa_i(m_i^+) - \sum_i \kappa_i(m_i^-) - \sum_{i,j}\left(\phi_{ij}\lambda_{ij}^-\right)(m_i^+ - m_i^-) + \sum_{i,j}\left(x_{ij}^+ - x_{ij}^-\right)\lambda_{ij}^- = \begin{cases} 0 & \text{if } t \in T^e \\ \rho\left(\dot{y} - f_1\dot{l}_1^+ - f_2\dot{l}_2^+\right) & \text{if } t \in T^l \end{cases} \tag{5.18}$$

Or, the same relation can be expressed as

$$\sum_i \kappa(m^-) - \sum_i \kappa(m^+) - \sum_{i,j}\left(\phi_{ij}\lambda_{ij}^-\right)(m^- - m^+) + \sum_{i,j}\left(x_{ij}^- - x_{ij}^+\right)\lambda_{ij}^- = \begin{cases} \rho\left(\dot{y} - f_1\dot{l}_1^- - f_2\dot{l}_2^-\right) & \text{if } t \in T^e \\ 0 & \text{if } t \in T^l \end{cases} \tag{5.18'}$$

Proposition 15. *At both entering and leaving points, m_i and λ_{ij} are continuous at $m_i = \bar{m}_i$ and at $\lambda_{ij} \geq 0$. x_{ij} is continuous at $x_{ij} = 0$ both at entering and leaving points when $\dot{y} \geq -\sum_{i,j}\sigma_{ij}l_{ij}$. If $\dot{y} < -\sum_{i,j}\sigma_{ij}l_{ij}$, $x_{ij}^+ > 0$ for some (i, j) showing discontinuity at entering points. Also, entering time is characterized by*

$$\kappa'\left(\bar{m}_i(t^e)\right) = \sum_j \phi_{ij}\lambda_{ij}^-(t^e) \tag{5.19}$$

where $t^e \in T^e$.

Proof. When $t \in T^e$, $\lambda_{ij}^- \geq 0$ for $\forall i, j$. If $\lambda_{ij}^- < 0$ for some i and j, then $x_{ij}^- = X$, which implies $\sum_{i,j} f_{ij}\dot{l}_{ij}^- \ll \dot{y}$, violating the entering condition $\sum_{i,j} f_{ij}\dot{l}_{ij}^- > \dot{y}$. First, suppose $\lambda_{ij}^- > 0$ for some i and j. Then, (5.18) becomes

$$\sum_i \kappa_i(m_i^+) = \sum_i \kappa_i(m_i^-) + \sum_i \kappa_i'(m^-)(m_i^+ - m_i^-) - \sum_{i,j} x_{ij}^+\lambda_{ij}^-$$

when $t \in T^e$. However, since κ_i is a convex function and $x \geq 0$, the above relation is only possible when $m_i^+ = m_i^-$ and $x_{ij}^+ = 0$ for all i and j. On the other hand, if $\lambda_{ij}^- = 0$ for all i and j, (5.18) yields $\sum_i \kappa_i(m_i^+) = \sum_i \kappa_i(m_i^-) = 0$ and again m_i is continuous at zero for all i. In this case, $k_{ab}^i(t^e) = \lambda_{ia}^+/f_{ia} - \lambda_{ib}^+/f_{ib} = \lambda_{ia}^-/f_{ia} - \lambda_{ib}^-/f_{ib} = 0$ for all i, a, b which implies $x_{ij} = 0$ for all i, j as far as $\dot{y} \geq -\sum_{i,j}\sigma_{ij}l_{ij}$ by Proposition 13. If $\dot{y} < -\sum_{i,j}\sigma_{ij}l_{ij}$, some of x_{ij} are strictly positive according to (2) of Proposition 13. Setting $m_i^+ = m_i^-$ and $x_{ij}^+ = x_{ij}^- = 0$ in (5.18') gives $\rho = 0$ which implies that λ_{ij} is continuous for all i, j at $\lambda_{ij} = \lambda_{ij}^- \geq 0$ from (5.15). At leaving points, (5.18') becomes

$$\sum_i \kappa_i(m_i^-) = \sum_i \kappa_i(m_i^+) + \sum_i \kappa_i'(m_i^+)(m_i^- - m_i^+) - \sum_{i,j} x_{ij}^-\lambda_{ij}^+$$

when $t \in T^l$, and since κ is a convex function and $x \geq 0$, the above relation is only possible when $m_i^+ = m_i^-$ and $x_{ij}^- = 0$ for all i and j. Putting these results in (5.18) gives us $\rho = 0$ which implies continuity of λ_{ij} for all i, j at leaving time. (5.20) comes from the fact that $\kappa_i'\left(m_i^-(t^e)\right) = \sum_j \phi_{ij}\lambda_{ij}^-(t^e)$ from (5.6), $m_i^+(t^e) = \bar{m}_i(t^e)$ from Proposition 13 and $m_i^-(t^e) = m_i^+(t^e)$ from the above results. □

From Proposition 15, entering points locate in domain $\sum_a \phi_{ia} f_{ia} k_{aj}^i \leq \kappa_i'(\bar{m}_i)$. It implies that the entering to the demand constraint must be "smooth" in the configuration space if $\dot{y} \geq -\sum_{i,j}\sigma_{ij}l_{ij}$. Namely, growth of labor must slow down as employment approaches to the demand constraint. This implies that the dynamics of employment immediately before entering is affected by the influence of m on the growth of each l_{ij}. The next proposition shows that the slowdown accompanies rotation in the configuration subspace in a simple way when there are only two different *undeclarable* types.

Proposition 16. *Suppose $L = 1$, $M_1 = 2$ and $\dot{y} = 0$. If the entering point is located below the line*

$$l_{12} = \frac{\phi_{12}\,\sigma_{11}}{\phi_{11}\,\sigma_{12}} l_{11},$$

the path prior to entering shows clockwise rotation, $\dot{l}_1(t^e) < 0$ and $\dot{l}_2(t^e) > 0$. If entering occurs below the line, it shows counterclockwise rotation, $\dot{l}_1(t^e) > 0$ and $\dot{l}_2(t^e) < 0$.

Proof. Denote $l_1 := l_{11}$ and $l_2 := l_{12}$ for simple notation. From Proposition 15, $m(t^e)$ is continuous at $m(t^e) = \bar{m} = (\sigma_1 f_1 l_1 + \sigma_2 f_2 l_2)/(\phi_1 f_1 + \phi_2 f_2)$. The slope of displacement vector immediately prior to entering is given by

$$\left.\frac{dl_2}{dl_1}\right|_{t\uparrow t^e} = \frac{\phi_2\bar{m} - \sigma_2 l_2}{\phi_1\bar{m} - \sigma_1 l_1} = -\frac{f_1}{f_2}$$

Its time derivative becomes

$$\frac{d}{dt}\left(\frac{dl_2}{dl_1}\right) = -\frac{f_1}{f_2}\left(\frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2}\right)$$

Then,

$$\begin{aligned} \frac{d}{dt}\left(\frac{dl_2}{dl_1}\right) \gtreqless 0 \quad &\Longleftrightarrow \quad \frac{\dot{f}_1}{f_1} \lesseqgtr \frac{\dot{f}_2}{f_2} \\ &\Longleftrightarrow \quad \frac{\dot{l}_1}{f_2}\begin{pmatrix} f_2 & f_1 \end{pmatrix}\begin{bmatrix} f_{11} & -f_{12} \\ -f_{21} & f_{22} \end{bmatrix}\begin{pmatrix} f_2 \\ f_1 \end{pmatrix} \lesseqgtr 0 \end{aligned}$$

Note that the matrix in the last line has common principle minors as $\nabla^2 f(\boldsymbol{l})$. Thus, from the concavity of f, the quadratic form part is positive. Thus, the sign of $\dot{l}_1$ coincides with the sign of the left hand side of the last line. From $\dot{l}_1 \lesseqgtr 0 \Leftrightarrow l_2/l_1 \lesseqgtr (\phi_2\,\sigma_1)/(\phi_1\,\sigma_2)$, we obtain

$$\frac{d}{dt}\left(\frac{dl_2}{dl_1}\right) \gtreqless 0 \quad \Longleftrightarrow \quad \frac{l_2}{l_1} \lesseqgtr \frac{\phi_2}{\phi_1}\frac{\sigma_1}{\sigma_2}.$$

Also, note $\dot{l}_1(t^e) \lesseqgtr 0 \Leftrightarrow \dot{l}_2(t^e) \gtreqless 0$, thus the proposition is derived. □

The path behavior depicted in Proposition 16 is shown in Figure 7.1 on page 32. The behavior with more than three types or with different *declarable* labor types prior to entering is more complex. In the latter case, the displacement vector is characterized by $dl^-_{ia}/dl^-_{jb} = (\phi_{ia}m_i - \sigma_{ia}l_{ia})/(\phi_{jb}m_j - \sigma_{jb}l_{jb})$ the movement of which is also affected by the relative convergence speed of m_i and m_j, not only relative size of labor.

Leaving condition from the demand constraint. Leaving points exist in the *interior* of $\sum_a \phi_{ia} f_{ia} k^i_{aj} \le \kappa'_i(\bar{m}_i)$. Note that leaving from the demand constraint never occurs so far as $\dot{y} \le 0$. Leaving occurs when catchup to the growth of demand becomes too costly in terms of accompanying vacancy cost. As $\dot{y}$ becomes too large, it becomes suboptimal to stick to the surface of the demand constraint. What happens is that as $\dot{y}$ grows, the band $B \geq k \geq C$ ($\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa_i'(\bar{m}_i)$) in Figure 5.1 widens while the width of other bands are kept constant. It implies that for given k, it becomes more likely to fall in the domain $B \geq k \geq C$. Unless the value of $\dot{y}$ is such that corresponding optimal control keeps the state variables *exactly* on the surface of the demand constraint, as soon as k falls in the domain $B \geq k \geq C$, the state variables leaves the demand constraint. The leaving is more likely to happen if $|k|$ is small.

Proposition 17. *Leaving time is characterized by*

$$\kappa'\left(\bar{m}_i(t^l)\right) = \sum_j \phi_{ij}\lambda^+_{ij}(t^l) \tag{5.20}$$

where $t^l \in T^l$. *Leaving points satisfy the condition* $\sum_a \phi_{ia} f_{ia} k^i_{aj} < \kappa_i'(\bar{m}_i)$.

Proof. From Proposition 17 and $\dot{y}(t^l) > 0$, m and x are continuous at t^l and $x_{ij}(t^l) = 0$ for all i, j, from which (5.20) is derived. From Proposition 17, $\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa_i'(\bar{m}_i)$ must hold at $t = t^l$. Also, from (5.6) and $\dot{y}(t^l) > 0$, $\lambda^+_{ij} > 0$ must hold for some i, j. Suppose $\sum_a \phi_{ia} f_{ia} k^i_{aj} = \kappa_i'(\bar{m}_i)$. Then, for any i, j, $\sum_a \phi_{ia} f_{ia}\left(\lambda_{ia}/f_{ia} - \lambda_{ij}/f_{ij}\right) = \sum_a \phi_{ia}\lambda_{ia}$ holds which implies $\lambda_{ij} = 0$ for any i, j. This is a contradiction. □

6. A Note on the Concavity of Coalitional Benefits

The wage functions derived in Section 3, 4 and Appendix C are supported by Shapley value with the assumptions given so far. However, for them to be supported additionally by nucleolus, concavity of coalitional benefit of the entrepreneur and workers F is required, which is given by

$$F(\boldsymbol{l}_t) = \int_t^\infty \left(f - \sum_i \kappa_i\right) e^{-\int r} d\xi$$

where $\boldsymbol{l}$ follows the optimal employment path for the firm and the above equation is evaluated off the demand constraint because it relates to the case some of agents are hypothetically leave the coalition. Its instantaneous part is concave, thus as far as the dynamics of states are relatively uniform over the state space, which in turn implies uniformity in curvature of f, one can expect that $F(\cdot)$ is concave. However, there can be a pathological case in which the path starting from an internally dividing point of two given initial states goes too far so that concavity of the instantaneous part as a function of the initial points is violated. Denote the optimal path starting from $\boldsymbol{l}^0$ by $\boldsymbol{l}(\boldsymbol{l}^0, t')$ where $t' \geq t$ denotes time. Then, $\boldsymbol{l}(\alpha\boldsymbol{l}^0 + (1-\alpha)\boldsymbol{l}^1, t')$ can go too far away from $\boldsymbol{l}(\boldsymbol{l}^0, t')$ and $\boldsymbol{l}(\boldsymbol{l}^1, t')$ at some time t' so that $(f - \sum \kappa_i)\big|_{\boldsymbol{l}=\boldsymbol{l}(\alpha\boldsymbol{l}^0+(1-\alpha)\boldsymbol{l}^1,t')} < \alpha(f - \sum \kappa_i)\big|_{\boldsymbol{l}=\boldsymbol{l}(\boldsymbol{l}^0,t')} + (1-\alpha)(f - \sum \kappa_i)\big|_{\boldsymbol{l}=\boldsymbol{l}(\boldsymbol{l}^1,t')}$ becomes to hold where all variables are evaluated on the optimal paths. The optimal dynamics is given by

$$\dot{\boldsymbol{l}}_i = \boldsymbol{\phi}_i \cdot \boldsymbol{m}\left(\lambda(\boldsymbol{l})\right) - \boldsymbol{\sigma}_i \cdot \boldsymbol{l}_i$$

where the linear part preserves the internally dividing point being the internally dividing point of two end points at any time, so the pathology comes from the degree of non-linearity in the first term. From (5.6),

$$m_i = \kappa'^{-1}\left(\boldsymbol{\phi}_i \cdot \boldsymbol{\lambda}_i\right)$$

on the optimal path, which shows m_i is a positive function of $\boldsymbol{\phi}_i \cdot \boldsymbol{\lambda}_i$, and

$$\lambda_{ij}(t) \quad = \quad \int_t^{t^e} \left(\frac{\partial f}{\partial l_{ij}} - \sum_{k=1}^{L} \frac{\partial c_{ik}}{\partial l_{ij}} \right) e^{-\int (r+\sigma_{ij})} d\xi + C_{ij}(t^e) \tag{6.1}$$

where t^e is the entering time to the demand constraint. It shows the uniformity in curvature of f is required to avoid such a pathological case.

7. Steady State on the Demand Surface

The model allows for analysis of a perpetually moving economy by truncating the economy in sufficiently distant future. However, to settle down the endpoint of costate variables, it is sometimes convenient to analyze the steady state. The previous analyses showed that unless there is coordinated expectation among economic agents which persists for infinite length of time, the economy will not reach to the unbounded steady state. On the other hand, the economy can be settled in a steady state on the stationary demand constraint. Suppose $\dot{y} = 0$ in this section. Then, we find out strictly positive amount of rejection of job application at steady state "almost surely". A bounded steady state maximizes profits obtainable when initial state of labor can be directly chosen. Consider the following static problem.

$$\max_{l,m,x} \left\{ f(\boldsymbol{l}) - w(\boldsymbol{l}) \cdot \boldsymbol{l} - \sum_{i=1}^{L} \kappa_i(m_i) \right\} \tag{P'}$$

subject to

$$\phi_{ij} m_i = \left(r + \sigma_{ij}\right) l_{ij} + x_{ij}, \ \forall i, j \tag{2.1'}$$

$$y = f(\boldsymbol{l}) \tag{5.3'}$$

Theorem 18. *Steady-state solution of problem (P) is equivalent to the solution of (P').*

Proof. The optimality condition of the problem (P') is given by

$$\kappa_i'(m_i) = \sum_j \phi_{ij} \hat{\lambda}_{ij} \tag{5.6'}$$

$$\hat{\lambda}_{ij} = \frac{f_{ij} - c_{ij}}{r + \sigma_{ij}} - \hat{\mu}_0 \frac{f_{ij}}{r + \sigma_{ij}} \tag{5.9'}$$

$$x_{ij} = \begin{cases} 0 & \text{if } \hat{\lambda}_{ij} > 0 \\ X & \text{if } \hat{\lambda}_{ij} < 0 \end{cases} \tag{5.10'}$$

and the constraints where $\hat{\lambda}_{ij}$ and $\hat{\mu}_0$ are costate variables adjoint to equations (2.1') and (5.3'), respectively. From (5.6') and (5.9'),

$$\hat{\mu}_0 = \frac{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}}(f_{ij} - c_{ij}) - \kappa_i'(m_i)}{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}} f_{ij}} \tag{7.1}$$

Since X is arbitrarily large and therefore the steady state condition for l_{ij} does not hold when $x_{ij} = X$, $\hat{\lambda}_{ij} < 0$ is impossible for all i. Thus, $\hat{\lambda}_{ij} > 0$ or $\hat{\lambda}_{ij} = 0$. If there exist (i, j) such that $\hat{\lambda}_{ij} = 0$, then for such (i, j)'s

$$\hat{\mu}_0 = \frac{f_{ij} - c_{ij}}{f_{ij}} \qquad \forall (i, j),\ \hat{\lambda}_{ij} = 0 \tag{7.2}$$

and for other (i, j)'s such that $\hat{\lambda}_{ij} > 0$,

$$x_{ij} = 0 \qquad \forall (i, j),\ \hat{\lambda}_{ij} > 0 \tag{7.3}$$

holds. Then, the solution is completely characterized by (2.1'), (5.3'), (7.1), (7.2) and (7.3).

On the other hand, the bounded steady state solution to the original problem (P) is given by imposing steady state condition $\dot{l} = \dot{\lambda} = \dot{f}_{ij} = 0$ to each optimal condition. Imposing it on (2.1) and (5.3) obtains the same condition as (2.1') and (5.3'). From (5.9) and the steady state conditions,

$$\lambda_{ij} = \frac{f_{ij} - c_{ij}}{r + \sigma_{ij}} + \mu_0 \frac{\sigma_{ij} f_{ij}}{r + \sigma_{ij}}. \tag{5.9''}$$

Substituting this to (5.6) derives

$$\mu_0 = \frac{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}}(f_{ij} - c_{ij}) - \kappa_i'(m_i)}{r \sum_j \frac{\phi_{ij}}{r+\sigma_{ij}} f_{ij}}, \tag{7.1'}$$

which is equivalent to (7.1) if we define $\hat{\mu}_0 = r\mu_0$. From (5.9') and (5.9''), $\lambda_{ij} = \hat{\lambda}_{ij} + \mu_0 f_{ij}$, which results in equivalence relation between

$$x_{ij} = \begin{cases} 0 & \text{if } A_{ij} > 0 \\ [0, X] & \text{if } A_{ij} = 0 \end{cases} \text{ in problem (P)} \quad \Longleftrightarrow \quad x_{ij} = \begin{cases} 0 & \text{if } \hat{\lambda}_{ij} > 0 \\ [0, X] & \text{if } \hat{\lambda}_{ij} = 0 \end{cases} \text{ in problem (P').}$$

All of the above equivalences show that problem (P') is equivalent to problem (P). □

The next theorem shows that, in general, the point in which long-run profit is maximized does not coincide with the point in which a bounded steady state is achieved with no-firing. It means that either dismissal or rejection of job application will occur at a bounded steady state.

Theorem 19. *If* $\max_i M_i \geq 2$, *the set of parameters* (ϕ, σ) *that brings* $x_{ij} = 0$ *for all* (i, j) *at steady state has measure zero in the parameter space for given* f *and* κ.

Proof. From Theorem 18, the proposition can be proved via problem (P'). x which appears in (P') can be viewed as a slack variable substituting equality of equation (2.1') with inequality. Namely, it is equivalent to the following problem:

$$\min_{l,m}\left\{w(\boldsymbol{l})\cdot\boldsymbol{l}+\sum_{i=1}^{L}\kappa_i(m_i)\right\} \tag{P''}$$

subject to

$$\phi_{ij}m_i \geq \left(r+\sigma_{ij}\right)l_{ij}, \quad \forall i, j \tag{2.1''}$$

$$y = f(\boldsymbol{l}) \tag{5.3'}$$

Obviously, m_i maximizes the maximand when it is set to $m_i = \min_j\{(r+\sigma_{ij})l_{ij}/\phi_{ij}\}$ in equation (2.1"). Maximization on $\boldsymbol{l}$ with this condition completely determines solution for $\boldsymbol{l}$. However, in general,

$$\frac{r+\sigma_{ij}}{\phi_{ij}}l_{ij} \neq \frac{r+\sigma_{ij'}}{\phi_{ij'}}l_{ij'}$$

for any $j' \neq j$, making $x_{ij'} > 0$ for any j' such that $j' \neq \arg\min_j\{(r+\sigma_{ij})l_{ij}/\phi_{ij}\}$. Even when the condition

$$\frac{r+\sigma_{ij}}{\phi_{ij}}l_{ij} = \frac{r+\sigma_{ij'}}{\phi_{ij'}}l_{ij'} \tag{7.4}$$

for all j, j', i holds, it fails to hold once any small perturbation is added on one of r,σ or ϕ keeping other parameters. Namely, a set of parameters which satisfies (7.4) does not contain inner points, which implies that it has zero measure in the parameter space when $\sum_{i=1}^{L} M_i \geq 2$. □

The above theorem shows that dismissal or rejection of application *generically* occurs at least in one of the labor types not only in transition on the demand constraint surface but also at steady state, when there exist more than two labor types in the economy.

Figure 7.1 shows typical dynamics toward steady state when $L = 1$ and $M_1 = 2$. Paths starting from initial points A_1 and A_2 converge to a steady state C via entering points B_1 and B_2, respectively, when steady state demand level is y_1. The entering to the demand surface shows clockwise rotation above line OP and counterclockwise rotation below OP as shown at B_1 and B_2. If the demand shifts up unexpectedly to $y = y_2$ in neighborhood of C, the path starts to move toward the new demand surface and after counterclockwise entering, it converges to the new bounded steady state D.

8. A Note on Demand for Working Capital and the Rate of Interest

One of characteristics peculiar to search models is that firms are required to put "advances". Especially, when $\boldsymbol{l} = 0$, they must find a way to finance those advances, as the classicals such as Quesnay and Ricardo used to assume. Let us assume firms demand for working capital for this reason. Since λ is higher when l is low, the demand for lending is higher for a smaller firm with unbounded demand. On the other hand, at a bounded steady

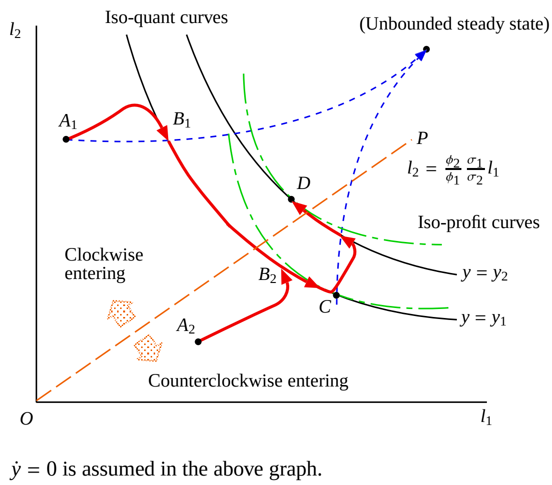


Figure 7.1: Typical trajectories and effect of unexpected shift of demand constraint

state, required m and therefore necessary working capital $\kappa(m)$ are higher when y is higher. This observation provides two contradicting tendencies depending on whether or not the demand constraint is binding. Suppose that firms are homogeneous and supply of working capital is constant. If the demand constraint is unbinding in the economy, interest rate gradually decreases as the economy grows. On the other hand, comparing two economies staying at bounded steady states with different level of y, the rate of interest is higher for the developed economy than the other. This fact may explain so-called allocation paradox (Lucas (1990); Gourinchas and Jeanne (2007)). Even though our model did not introduce physical capital, if capital should be interpreted as fund to cover the set-up cost, the same logic can be applied in an extended model.[19] Under the presence of friction, the state of coordinated expectation critically affects the equilibrium rate of interest.

9. Concluding Remarks

This paper showed that if there is search friction representable by a convex vacancy cost function ——however small for a given amount of hiring——, the economy obeys the effective demand principle. Wage rate is always smaller than marginal productivity, and a direct attempt to lower wage rate will not remove unemployment, as the old Keynesian arguments suggest. It should be noted that any kind of sticky prices is not assumed in this model. The existence of convex vacancy cost prohibits convergence to an unbounded steady state, or an *equilibrium in the long run*, without persistent coordination of expectation. Wage rate is flexible reflecting redundant resources in the labor market. One of major consequences of the search theory is that, when search friction is present, Keynes's first postulate of classicals —the wage is equal to the marginal product of labour— must be abandoned on a rational basis, but also the present paper showed that, together with a convex vacancy cost, the following second postulate must be modified from the literal sense: the utility of wages when a given volume of labour is employed is equal to the marginal disutility of that amount of employment. Instead, the second postulate is maintained in a broader

[19] If there is no genuine working capital, those financial and physical capitals coincide.

sense that workers follow their optimal choice, only on a boundary. Workers' optimal behavior is to work more below unbounded steady states, but such behavior is bounded by limited working opportunities. Anyway, this partial rejection of the second postulate enables involuntary unemployment — not by secular interpretation, but in the original definition: *men are involuntarily unemployed if, in the event of a small rise in the price of wage-goods relatively to the money-wage, both the aggregate supply of labour willing to work for the current money-wage and the aggregate demand for it at that wage would be greater than the existing volume of employment* (Keynes (1936, p.15)). Since wage bargaining is based on rational expectation on both sides, there is no built-in mechanism which brings the economy back to a natural level of output nor natural rate of unemployment. If there is a tendency toward full employment, it must be pursued in exogenous factors from the model presented here. One of important factors excluded from our model is the possibility for workers to escape to autarky. This can affect long-run unemployment rate and can be a source of poverty trap.

In search models, profit of a firm is strictly positive even when the commodity market is competitive. The fact that an entrepreneur earns non-zero profit and that he has massive power in bargaining as suggested in this paper raises a fundamental question that who really is the "entrepreneur". The question cannot be neglected when one undertakes explicit specification of the demand side since it affects the distribution of income and potentially the level of investment. There can be two most straightforward but extreme ways of extension: one is to assume that income level has no impact on pattern of consumption and investment. The other is to assume that there are two classes, haves and have-nots in the Kaldorian way. The latter literally assumes that the entrepreneur (and his successor) embodies *all* the knowledge necessary for firm's management and that it is never transferred to workers. However, as many examples show, even family successors must learn management as workers before he succeeds the company. This fact shows that more complicated internal forces are working in firms' organization.

Appendix A. Distribution of Coefficients of (3.3)

Proposition 20. *Let* $(\zeta_1, \ldots, \zeta_N) \in \mathbb{N}^N$ *be a vector of parameters. For any* $y_i \in \mathbb{N}$ *such that* $0 \le y_i \le \zeta_i$*, define*

$$\Upsilon(y_1, \ldots, y_N; \zeta_1, \ldots, \zeta_N) := \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\prod_{i=1}^M \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \tag{A.1}$$

Then, equation (A.1) *is a probability mass function.*

Proof. $\Upsilon \ge 0$ is obvious. If we sum it up for all x_i, it becomes

$$\begin{aligned}
\sum \Upsilon &= \frac{1}{1 + \sum_{i=1}^N \zeta_i} \sum_{y_1=0}^{\zeta_1} \cdots \sum_{y_M=0}^{\zeta_M} \frac{\prod_{i=1}^N \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^N \zeta_i}{\sum_{i=1}^N y_i}} \\
&= \frac{1}{1 + \sum_{i=1}^N \zeta_i} \sum_{k=0}^{\sum \zeta_i} \sum_{\{y_i : \sum_{i=1}^N y_i = k\}} \frac{\prod_{i=1}^N \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^N \zeta_i}{k}} \\
&= \frac{1}{1 + \sum_{i=1}^N \zeta_i} \sum_{k=0}^{\sum \zeta_i} \sum_{\sum y_i = k} \text{Mult.Hypg.}(y_1, \ldots, y_N; k; \zeta_1, \ldots, \zeta_N)
\end{aligned}$$

$$= \quad 1$$

where Mult.Hypg.$(y_1, \ldots, y_N; k; \zeta_1, \ldots, \zeta_N)$ is a multivariate hypergeometric distribution with parameter $(k; \zeta_1, \ldots, \zeta_N)$. It sums up to one if all n_i's are summed up keeping $\sum n_i = k$. □

Proposition 21. *Define a density function $\tilde{\Upsilon} : \mathbb{R}^N \to \mathbb{R}$ characterized by Υ such that*

$$\tilde{\Upsilon}(x_1, \ldots, x_N)\, dl_1 \cdots dl_N = \Upsilon(y_1, \ldots, y_N) \tag{A.2}$$

where $x_i = y_i dl_i$ and $0 \le x_i \le l_i$ where l_i is fixed for any ζ_i and dl_i keeping $l_i = \zeta_i dl_i$ ($i = 1, \ldots, N$). Then, the functional form of $\tilde{\Upsilon}$ is given by

$$\tilde{\Upsilon}(x_1, \ldots, x_N) = \delta\left(1 - \frac{x_1}{l_1}, \ldots, 1 - \frac{x_N}{l_N}\right)$$

as $\zeta_i \to \infty$ for all i where δ denotes Dirac's delta, i.e.

$$\delta(z_1, \ldots, z_N) = \begin{cases} \infty & \text{if } \forall i,\ z_i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_0^1 \cdots \int_0^1 \delta(z_1, \ldots, z_N)\, dz_1 \cdots dz_N = 1. \tag{A.3}$$

Proof. From Proposition 20,

$$\sum_{y_1=1}^{\zeta_1} \cdots \sum_{y_N=1}^{\zeta_N} \Upsilon(y_1, \ldots, y_N) = 1.$$

Using (A.2), it means

$$\sum_{x_1=dl_1}^{l_1} \cdots \sum_{x_N=dl_N}^{l_N} \tilde{\Upsilon}(x_1, \ldots, x_N)\, dl_1 \cdots dl_N = 1$$

which leads to show $\tilde{\Upsilon}$ satisfies property (A.3) as $\zeta_i \to \infty$, i.e. $dl_i \to 0$, for all i. Note that

$$\left(\prod_{i=1}^M \zeta_i\right)\left(\prod_{i=1}^M \binom{\zeta_i}{y_i}\right) = o\left(\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}\right)$$

if there exists i such that $y_i < \zeta_i$. Then, (A.2) becomes

$$\begin{aligned} \tilde{\Upsilon}(l_1, \ldots, l_N) &= \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\prod_{i=1}^M \binom{\zeta_i}{y_i}}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \frac{1}{dl_1 \cdots dl_N} \\ &= \frac{1}{\prod_{i=1}^N l_i} \frac{1}{1 + \sum_{i=1}^N \zeta_i} \frac{\left(\prod_{i=1}^N \zeta_i\right)\left(\prod_{i=1}^M \binom{\zeta_i}{y_i}\right)}{\binom{\sum_{i=1}^M \zeta_i}{\sum_{i=1}^M y_i}} \to 0 \end{aligned}$$

as $\zeta_i \to \infty$ for all i if there exists i such that $y_i < \zeta_i$. On the other hand, if $y_i = \zeta_i$ for all i, we have $\prod_{i=1}^M \binom{\zeta_i}{y_i} = 1$ and thus

$$\Upsilon(\zeta_1, \ldots, \zeta_N) = \frac{1}{1 + \sum_{i=1}^N \zeta_i}.$$

Then, from (A.2),

$$\begin{aligned}\tilde{\Upsilon}(l_1,\ldots,l_N) &= \frac{1}{1+\sum_{i=1}^N \zeta_i}\frac{1}{dl_1\cdots dl_N}\\ &= \frac{1}{\prod_{i=1}^N l_i}\frac{\prod_{i=1}^N \zeta_i}{1+\sum_{i=1}^N \zeta_i}.\end{aligned}$$

The second fraction diverges as ζ_i's become large, therefore $\tilde{\Upsilon}(l_1,\ldots,l_N)\to\infty$ as $\zeta_i\to\infty$ for all i. □

Appendix B. A Property of Essentially Concave Game

Below is the proof of Lemma 3 in section 3.

Lemma 22 (Lemma 3)**.** *If game (Ω, v) is essentially concave in which players are partitioned by groups such that $\Omega=\bigcup_{i=1}^M S_i$ and $\bigcap_{i=1}^M S_i=\emptyset$, then for any $S,T\subseteq\Omega$ such that $S\subset T$, the following inequality holds.*

$$v(T)-v(T\setminus S)\geq\sum_{i=1}^n \|S\cap S_i\|\Big[v(T)-v\Big(T\setminus\{s_i(j)\}\Big)\Big]$$

Proof. We use the fact that $v(T)-v(T\setminus S)$ has common value regardless of how players of S are removed from T. Define $\mathfrak{S}_{i_1 i_2\cdots i_m}(n_{i_1},\ldots,n_{i_m}):=\bigcap_{k=\{1,\ldots,m:n_{i_k}\neq 0\}}\bigcap_{j=1}^{n_{i_k}}\Big\{s_{i_k}(j)\Big\}$ where $1\leq m\leq M$ and $1\leq n_i\leq N_i$. When $n_{i_k}=0$ for all k, define $\mathfrak{S}_{i_1 i_2\cdots i_m}(0\cdots 0)=\emptyset$ for convenience. Then, for given $k\in\{1,\ldots,M\}$,

$$\begin{aligned}&v(T)-v(T\setminus S)\\ &= \sum_{n_k=1}^{N_k}\Big[v\Big(T\setminus\mathfrak{S}_k(n_k-1)\Big)-v\Big(T\setminus\mathfrak{S}_k(n_k)\Big)\Big]+\sum_{n_{i_2}=1}^{N_{i_2}}\Big[v\Big(T\setminus\mathfrak{S}_{ki_2}(N_k,\,n_{i_2}-1)\Big)-v\Big(T\setminus\mathfrak{S}_{ki_2}(N_k,n_{i_2})\Big)\Big]\\ &\quad+\cdots+\sum_{n_{i_M}=1}^{N_{i_M}}\Big[v\Big(T\setminus\mathfrak{S}_{ki_2\cdots i_M}(N_k,\ldots,N_{i_{M-1}},\,n_{i_M}-1)\Big)-v\Big(T\setminus\mathfrak{S}_{ki_2\cdots i_M}(N_1,\ldots,N_{i_{M-1}},n_{i_M})\Big)\Big]\\ &\geq M\sum_{n_k=1}^{N_k}\Big[v\Big(T\setminus\mathfrak{S}_k(n_k-1)\Big)-v\Big(T\setminus\mathfrak{S}_k(n_k)\Big)\Big]\end{aligned}$$

where $i_2,\ldots,i_M$ are taken in an arbitrary order so that $i_j\neq k$ and $i_j\neq i_{j'}$ if $j\neq j'$. The last line comes from the essential concavity. Summing up the above inequality for all $k=1,\ldots,M$,

$$\begin{aligned}v(T)-v(T\setminus S) &\geq \sum_{k=1}^{M}\sum_{n_k=1}^{N_k}\Big[v\Big(T\setminus\mathfrak{S}_k(n_k-1)\Big)-v\Big(T\setminus\mathfrak{S}_k(n_k)\Big)\Big]\\ &\geq \sum_{k=1}^{M}\|S\cap S_k\|\Big[v(T)-v\Big(T\setminus\Big\{s_k(j)\Big\}\Big)\Big]\end{aligned}$$

for any $j=1,\ldots,N_k$ again from the essential concavity. □

Appendix C. Wage Function in a General case

In this section, the case is handled in which worker separation rates are not common for all type of workers. The following approach can be used not only to obtain an explicit functional form but also for numerical calculations.

The Bellman equations (2.4) and (2.5) are equivalent to

$$\begin{aligned} U_i(t) &= \mathsf{E}_\xi \left[\int_t^\xi b_i(\tau)\, e^{-\int_t^\tau r}\, d\tau + \mathsf{E}_j E_{ij}(\xi)\, e^{-\int_t^\xi r} \right] \\ E_{ij}(t) &= \mathsf{E}_\xi \left[\int_t^\xi w_{ij}(\tau)\, e^{-\int_t^\tau r}\, d\tau + U_i(\xi)\, e^{-\int_t^\xi r} \right] \end{aligned}$$

in integral forms, where E_ξ is an expectation operator on ξ. Using partial integration, they simplify to

$$U_i(t) = \int_t^\infty \Big(b_i(\xi) + \mu_i(\xi) \mathsf{E}_j E_{ij}(\xi) \Big) e^{-\int_t^\xi (r+\mu_i)}\, d\xi \tag{C.1}$$

$$E_{ij}(t) = \int_t^\infty \Big(w_{ij}(\xi) + \sigma_{ij}(\xi) U(\xi) \Big) e^{-\int_t^\xi (r+\sigma_{ij})}\, d\xi \tag{C.2}$$

where $\sigma_{ij}(\xi) \nrightarrow 0$ and $y(\xi) \nrightarrow 0$ as $\xi \to +\infty$ are assumed. If $\sigma_{ij} \nrightarrow 0$ and $y \nrightarrow 0$, then $\mu \nrightarrow 0$, since replacement demand for labor does not vanish. We leave r arbitrary but only assumed to be integrable. They guarantee existence of U and E. (C.1) and (C.2) are singular Volterra integral equations of the second kind and have a structure of

$$V_i(t) - \int_t^\infty K_i(t,\xi)\, V_i(\xi)\, d\xi = h_i(t) \tag{C.3}$$

where $V_i(t) = {}^t(U_i(t),\, E_{i1}(t), \ldots, E_{iM_i}(t))$, the *integral kernel* K_i is given by

$$K_i = \begin{pmatrix} K_{i00} & K_{i01} & \cdots & K_{i0M_i} \\ K_{i10} & K_{i11} & \cdots & K_{i1M_i} \\ \vdots & \vdots & \ddots & \vdots \\ K_{iM_i0} & K_{iM_i1} & \cdots & K_{iM_iM_i} \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & g_1 \mu_i(\xi)\, e^{-\int_t^\xi \mu_i} & \cdots & g_{M_i} \mu_i(\xi)\, e^{-\int_t^\xi \mu_i} \\ \hline \tilde{\sigma}_{i1}(\xi)\, e^{-\int_t^\xi \tilde{\sigma}_{i1}} & & & \\ \vdots & & O & \\ \tilde{\sigma}_{iM_i}(\xi)\, e^{-\int_t^\xi \tilde{\sigma}_{iM_i}} & & & \end{array} \right) e^{-\int_t^\xi r}$$

and the *exceptional part* h is given by[20]

$$h_i = \begin{pmatrix} h_{i0} \\ h_{i1} \\ \vdots \\ h_{iM_i} \end{pmatrix} = \begin{pmatrix} \int_t^\infty b_i(\xi)\, e^{-\int_t^\xi (r+\mu_i)}\, d\xi \\ \int_t^\infty w_{i1}(\xi)\, e^{-\int_t^\xi (r+\sigma_{i1})}\, d\xi \\ \vdots \\ \int_t^\infty w_{iM_i}(\xi)\, e^{-\int_t^\xi (r+\sigma_{iM_i})}\, d\xi \end{pmatrix}$$

The following proposition can be derived.

Proposition 23. *The solution to the simultaneous equations* (C.3) *is given by*

$$V_i(t) = h_i(t) + \int_t^\infty G_i(t,\xi)\, h_i(\xi)\, d\xi$$

where G_i is a Neumann series matrix in which $G_{ipq}(t,\xi) := \sum_{\zeta=1}^\infty \overset{*\,\zeta}{K}_{ipq}(t,\xi)$ $(p, q = 1, \ldots, M_i)$, provided that $G_{ipq}(t,\xi)$ uniformly converges.

[20]Subscript i is sometimes omitted below when obvious.

The composition of (0, 0)-kernel is given by

$$
\begin{aligned}
\overset{*}{K}{}^{2}_{00}(t,\xi) &= \sum_{\zeta=0}^{M_i} \left(K_{0\zeta} * K_{\zeta 0}\right)(t,\xi) = \sum_{\zeta=1}^{M_i} \left(K_{0\zeta} * K_{\zeta 0}\right)(t,\xi) \\
&= e^{-\int_t^\xi r} \int_t^\xi \mu(\tau)\, e^{-\int_t^\tau \mu_i} \left(\sum_{\zeta=1}^{M_i} g_{i\zeta} \tilde{\sigma}_{i\zeta}(\xi)\, e^{-\int_\tau^\xi \tilde{\sigma}_{i\zeta}} \right) d\tau
\end{aligned}
$$

The integral part equals the expected probability that an unemployed worker as of time t is employed afterwards and separates again exactly at time ξ where expectation is taken for possible undeclarable types. Other cross-compositions are given by

$$
\begin{aligned}
&\overset{*}{K}{}^{2}_{0\zeta}(t,\xi) = \overset{*}{K}{}^{2}_{\zeta 0} = 0, \qquad \forall \zeta = 1,\dots,M_i \\
&\overset{*}{K}{}^{2}_{pq}(t,\xi) = \int_t^\xi K_{p0} K_{0q} = K_{p0} * K_{0q} \qquad \forall \{(p,q) \,|\, p \geq 1 \vee q \geq 1\}
\end{aligned}
$$

The iterated kernels alternate between zero and strictly positive numbers depending on whether the multiplicity of the iteration is odd or even. That is

$$
\begin{pmatrix}
\overset{*}{K}{}^{2n}_{00} & \cdots & \overset{*}{K}{}^{2n}_{0M_i} \\
\vdots & \ddots & \vdots \\
\overset{*}{K}{}^{2n}_{M_i 0} & \cdots & \overset{*}{K}{}^{2n}_{M_i M_i}
\end{pmatrix}
=
\left(
\begin{array}{c|ccc}
\overset{*}{K}{}^{2n}_{00} & 0 & \cdots & 0 \\
\hline
0 & & \vdots & \\
\vdots & \cdots & K_{p0} * \overset{*}{K}{}^{2n}_{00} * K_{0q} & \cdots \\
0 & & \vdots &
\end{array}
\right)
$$

and

$$
\begin{pmatrix}
\overset{*}{K}{}^{2n-1}_{00} & \cdots & \overset{*}{K}{}^{2n-1}_{0M_i} \\
\vdots & \ddots & \vdots \\
\overset{*}{K}{}^{2n-1}_{M_i 0} & \cdots & \overset{*}{K}{}^{2n-1}_{M_i M_i}
\end{pmatrix}
=
\left(
\begin{array}{c|ccc}
0 & \overset{*}{K}{}^{2(n-1)}_{00} * \overset{*}{K}_{01} & \cdots & \overset{*}{K}{}^{2(n-1)}_{00} * \overset{*}{K}_{0M_i} \\
\hline
K_{10} * \overset{*}{K}{}^{2(n-1)}_{00} & & & \\
\vdots & & O & \\
K_{M_i 0} * \overset{*}{K}{}^{2(n-1)}_{00} & & &
\end{array}
\right)
$$

for $n = 1, 2, \dots$ $\overset{*}{K}{}^{2n}_{00}$ comprises of the core part of iteration in each element and

$$
\overset{*}{K}{}^{2n}_{00}(t,\xi) = e^{-\int_t^\xi r} \int_t^\xi d\tau_{n-1} \int_t^{\tau_{n-1}} d\tau_{n-2} \cdots \int_t^{\tau_2} \left(\int_t^{\tau_1} A\right)\left(\int_{\tau_1}^{\tau_2} A\right) \cdots \left(\int_{\tau_{n-1}}^{\xi} A\right) d\tau_1
$$

where $A := e^{\int_t^\xi r} \overset{*}{K}{}^{2}_{00}$. The above can be interpreted as the expected discounted probability that an unemployed worker as of time t repeats the cycle of employment and separation n-times in the period of $(t, \xi]$ and the last separation occurs exactly at time ξ. Summing up the above results for all n, we obtain Neumann series G_{ij} =

$\sum_{n=1}^{\infty} \overset{*}{K}{}_{ij}^{n}$:

$$G = \left(\begin{array}{c|ccc} \sum_n \overset{*}{K}{}_{00}^{2n} & \sum_n \overset{*}{K}{}_{00}^{2(n-1)} * \overset{*}{K}_{01} & \cdots & \sum_n \overset{*}{K}{}_{00}^{2(n-1)} * \overset{*}{K}_{0M_i} \\ \hline \sum_n K_{01} * \overset{*}{K}{}_{00}^{2(n-1)} & & \vdots & \\ \vdots & \cdots & K_{p0} * \sum_n \overset{*}{K}{}_{00}^{2n} * K_{0q} & \cdots \\ \sum_n K_{M_i 0} * \overset{*}{K}{}_{00}^{2(n-1)} & & \vdots & \end{array}\right).$$

Using these results, the explicit form of wage rate function is given by the following proposition.

Proposition 24. *The wage rate of type-(i, j) worker as of time t is given by*

$$w_{ij}(t) = \omega_{ij}(t) + \sum_{k=1}^{M_i} \int_t^{\infty} \left(\sum_{n=1}^{\infty} \overset{*}{A}{}_{jk}^{n}(t, \xi) \right) e^{-\int (r+\sigma_{ij})} \omega_k(\xi)\, d\xi$$

where

$$\begin{aligned} A_{ij}(t, \xi) &:= \mu(t) \sum_{k=1}^{M_i} g_k \int_t^{\xi} \big(G_{kj}(t, \tau) - G_{0j}(t, \tau) \big) d\tau - \sigma_{ij}(t) \int_t^{\xi} G_{0j}(t, \tau) d\tau \\ \omega_{ij}(t) &:= \frac{\partial F(t)/\partial l_{ij} + b_i(t)}{2} - \frac{\mu_i(t) - \sigma_{ij}(t)}{2} \int_t^{\infty} b(\xi)\, e^{-\int (r+\mu_i)} d\xi \\ &\quad + \frac{1}{2} \int_t^{\infty} A_{j0}(t, \xi)\, b(\xi)\, e^{-\int (r+\mu_i)} d\xi \end{aligned}$$

for all i, j.

Proof. From Theorem 1, $\dot{E}_{ij} = \left[\dot{U}_i + \partial^2 F / (\partial t\, \partial l_{ij}) \right] / 2$. Substituting each value function by the time-derivative of the result of Proposition 23, we get

$$\text{(C.4)} \qquad \begin{aligned} w_{ij}(t) = {} & \frac{\partial F/\partial l_{ij}(t) + b_i(t)}{2} - \frac{\mu_i(t) - \sigma_{ij}(t)}{2} h_1 + \frac{1}{2} \int_t^{\infty} A_{j0}(t, \xi)\, b_i(\xi)\, e^{-\int (r+\mu_i)}\, d\xi \\ & + \frac{1}{2} \sum_{k=1}^{M_i} \int_t^{\infty} A_{jk}(t, \xi)\, w_k(\xi)\, e^{-\int (r+\sigma_{ij})} d\xi \end{aligned}$$

for all i, j. Solving simultaneous equation (C.4) obtains the result. □

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