The Chip Strategies Approximately Achieve Efficiency at the Optimal Rate*

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Abstract

For a two-player repeated favor-exchange game with private information, I compare the rates at which the chip-strategy equilibrium and the optimal perfect public equilibrium achieve the efficient payoff as the discount factor δ tends to 1. I show that (i) the convergence rate for the optimal perfect public equilibrium is no smaller than $(1-\delta)^{1/2}$, and (ii) that for the optimal chip-strategy equilibrium is no greater than $(1-\delta)^{1/2}$, where the number of total chips grows at rate $(1-\delta)^{-1/2}$. In this sense, the chip-strategy equilibrium approximately achieves efficiency at the optimal rate $(1-\delta)^{1/2}$.

Keywords: Repeated games, rates of convergence, chip strategies.
JEL codes: C72, C73.

1 Introduction

In the literature on repeated games, it is well known that the efficient payoffs are asymptotically achieved in equilibrium as the discount factor δ tends to 1 (e.g., Fudenberg and Maskin (1986), Fudenberg et al. (1994)). For games under imperfect monitoring, the asymptotic efficiency is often proved by using the self-generation technique of Abreu et al. (1990) rather than explicitly constructing equilibrium strategies.[1] By contrast,

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[1]For games with pure strategies, compact actions sets, and deterministic public signals, Laclau and Tomala (2017) characterize the equilibrium payoff set as δ reaches 1, by constructing equilibrium strategies based on cooperation, punishments, and rewards.

Olszewski and Safronov (2018) study specific equilibrium strategies by focusing on the chip strategies introduced by Möbius (2001), which represent a simple form of reciprocity. They show that chip-strategy equilibria asymptotically achieve the efficient payoff with a sufficiently large number of chips, in a class of symmetric repeated games with private information. As Abdulkadiroglu and Bagwell (2012, 2013) show, however, for some parameter values there are other equilibria that Pareto-dominate the chip-strategy equilibrium for a fixed discount factor.

In this paper, I study the performance of the chip-strategy equilibrium, relative to the optimal perfect public equilibrium, in terms of the convergence rate at which these equilibria achieve the efficient payoff as δ tends to 1. I consider a two-player favor-exchange game with private information, which is representative of the class of games considered by Olszewski and Safronov (2018). The main result of the current paper is that the chip-strategy equilibrium is optimal in the convergence rate. Precisely, I show that the efficiency loss for the chip-strategy equilibrium vanishes at the same rate as that for the optimal equilibrium, and the common rate of convergence is of order $(1-\delta)^{1/2}$. In this sense, the chip-strategy equilibrium approximately achieves the efficient payoff at the optimal rate.

The above result follows from two propositions. I show in Proposition 1 that the efficiency loss for the optimal perfect public equilibrium vanishes at rate at least $(1-\delta)^{1/2}$ (*lower bound*), and in Proposition 2 that the efficiency loss for the (optimal) chip-strategy equilibrium vanishes at rate at most $(1-\delta)^{1/2}$ (*upper bound*). Proposition 1 follows from a more general result, Theorem A.1, that in any repeated game under imperfect public monitoring, if the monitoring structure satisfies a certain weak form of full support condition, which holds in the favor-exchange game, an extremal payoff vector is reached by equilibria at rate at least $(1-\delta)^{1/2}$. To prove Proposition 2, I explicitly solve a recursive formula that characterizes the value of each chip holding, and obtain a closed form expression of the value function. This enables me to derive the divergence rate $(1-\delta)^{-1/2}$ of the optimal number of chips, and in turn the convergence rate of the efficiency loss for the chip-strategy equilibrium.

Favor-exchange games and chip strategies have been investigated in the literature on repeated games with private information. Möbius (2001) introduces and studies chip strategies in a two-player favor-exchange game with private information. He focuses on the chip strategy with only one chip and shows that it sustains cooperation on a network where agents interact only rarely with each other. As already mentioned earlier, Olszewski and Safronov (2018) show that the simple chip-strategy equilibria approximately achieve the efficient payoff as $\delta \to 1$ in a class of symmetric repeated games with private information. In particular, in their Proposition 1, they prove the asymptotic efficiency of the chip-strategy equilibrium in the favor-exchange game. In their proof, they apply the implicit function theorem to the recursive formula characterizing the value for each chip holding and obtain a first-order approximation of the value function, while I explicitly solve the formula and derive a closed form expression of the value function. For fixed discounting, it is known that the chip-strategy equilibrium is not optimal in general. Hauser and Hopenhayn (2008) present numerical examples in which the optimal equilibrium is strictly superior to the chip-strategy equilibrium. Abdulkadiroglu and Bagwell (2012, 2013) study more sophisticated favor-exchange equilibria where the size of a favor owed may decline over time, and show that these equilibria perform better than the simple chip-strategy equilibrium for some parameter values. The result of the current paper implies that these sophisticated equilibria do not improve the rate of convergence.

The rate of convergence in non-zero-sum discounted repeated games is first studied by Hörner and Takahashi (2016). They provide tight bounds on the rate of convergence of the equilibrium payoff sets for both the cases of perfect and imperfect monitoring. For the lower bound, they show in Proposition 4 that in a repeated prisoner's dilemma under imperfect public monitoring, the distance between the equilibrium payoff set and the full cooperation payoff vector vanishes at rate at least $(1-\delta)^{1/2}$. Theorem A.1 in the current paper generalizes their result as follows. First, Theorem A.1 applies to any repeated game under imperfect public monitoring beyond the prisoner's dilemma. Second, Theorem A.1 weakens the full support condition on the monitoring structure in Hörner and Takahashi (2016, Proposition 4) to accommodate the favor-exchange game, which does not satisfy their condition. For the upper bound, they show in their Proposition 8 that, for games under imperfect public monitoring, the distance between the equilibrium payoff set and a strictly individually rational payoff vector vanishes at rate at most $(1-\delta)^{1/2}$. They consider the entire set of perfect public equilibria and rely on the self-generation technique, while I focus on the specific class of chip strategies and obtain the same rate $(1-\delta)^{1/2}$ by using the closed form expression of the value function.

The paper is organized as follows. The rest of this section introduces the notations on the rate of convergence/divergence. Section 2 describes the favor-exchange model and the basic properties of the chip strategies. Section 3 states the results. Section 4 concludes the paper. Finally, the proofs of the results are presented in the Appendix.

Notation

I here introduce the notations for the rate of convergence/divergence. For a nonempty set $X \subset \mathbb{R}$, consider nonnegative-valued functions $f, g\colon X \to \mathbb{R}_+$. Let $\bar{x}$ be an arbitrary point in the closure of X. I write

- $f(x) = O(g(x))$ as $x \to \bar{x}$ if there exist $C > 0$ and $\varepsilon > 0$ such that $f(x) \leq Cg(x)$ for any $x \in X$ with $|x - \bar{x}| < \varepsilon$;
- $f(x) = \Omega(g(x))$ as $x \to \bar{x}$ if there exist $c > 0$ and $\varepsilon > 0$ such that $f(x) \geq cg(x)$ for any $x \in X$ with $|x - \bar{x}| < \varepsilon$; and
- $f(x) = \Theta(g(x))$ as $x \to \bar{x}$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$ as $x \to \bar{x}$.

When $\lim_{x\to\bar{x}} f(x) = \lim_{x\to\bar{x}} g(x) = 0$, I say that $f(x)$ vanishes at rate at most (resp. at rate at least, at the same rate as) $g(x)$ as $x \to \bar{x}$ if $f(x) = O(g(x))$ (resp. $\Omega(g(x))$, $\Theta(g(x))$) as $x \to \bar{x}$. When $\lim_{x\to\bar{x}} f(x) = \lim_{x\to\bar{x}} g(x) = \infty$, I say that $f(x)$ grows at rate at most (resp. at rate at least, at the same rate as) $g(x)$ if $f(x) = O(g(x))$ (resp. $\Omega(g(x))$, $\Theta(g(x))$) as $x \to \bar{x}$. I omit the phrase "as $x \to \bar{x}$" if $\bar{x}$ is evident from the context.

Let $\{a_n\}, \{b_n\}$ be sequences of nonnegative real numbers. I write

- $a_n = O(b_n)$ if there exist $C > 0$ and $N \in \mathbb{N}$ such that $a_n \leq Cb_n$ for all $n \geq N$;
- $a_n = \Omega(b_n)$ if there exist $c > 0$ and $N \in \mathbb{N}$ such that $a_n \geq cb_n$ for all $n \geq N$; and
- $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$.

When $\lim_n a_n = \lim_n b_n = 0$, I say that a_n vanishes at rate at most (resp. at rate at least, at the same rate as) b_n if $a_n = O(b_n)$ (resp. $\Omega(b_n)$, $\Theta(b_n)$). When $\lim_n a_n = \lim_n b_n = \infty$, I say that a_n grows at rate at most (resp. at rate at least, at the same rate as) b_n if $a_n = O(b_n)$ (resp. $\Omega(b_n)$, $\Theta(b_n)$).

2 Model

2.1 Favor-Exchange Model

I consider the two-player infinitely repeated favor-exchange game analyzed by Abdulkadiroglu and Bagwell (2012) and Olszewski and Safronov (2018, Section 2.1), which is a discrete time version of the model of Möbius (2001). In the stage game, either one of the following three events occurs independently across time: (i) only player 1 gets a good, (ii) only player 2 gets a good, and (iii) neither gets a good. The former two events occur with probability $p \in (0, 1/2)$ each, and the latter event occurs with the remaining probability $1 - 2p > 0$. Each player is privately informed whether he gets a good (state 1) or not (state 0). Thus, if a player does not receive a good, then he cannot observe whether his partner receives a good or not. For each $i \in \{1, 2\}$, let $Z_i := \{0, 1\}$ denote the set of privately observed states. If a player gets a good, he can consume the good (action 0) or give it to his partner (action 1). Otherwise, he has no action to choose. For each $i \in \{1, 2\}$, let $A_i := \{0, 1\}$ denote the set of state-contingent actions/stage-game strategies of player i. That is, "$a_i = 0$" (resp. "$a_i = 1$") mean that player i plans to consume the good (resp. give it to his partner) when he gets a good.

If a player consumes a good, then he gets a payoff of 1 and his partner gets a payoff of 0. If he gives the good to the partner, then the player gets a payoff of 0 and the partner gets a payoff of γ. Assume $\gamma > 1$, that is, the transfer is value-enhancing. For each $i \in \{1, 2\}$ and $a \in A$, where $A := A_1 \times A_2$, let $g_i(a)$ denote player i's expected payoff when players follow the action profile a;

$$g_i(a) = p(1 - a_i) + p\gamma a_j,$$

where $j \in \{1, 2\}$ and $j \neq i$. Define a function $g\colon A \to \mathbb{R}^2$ by $g(a) = (g_1(a), g_2(a))$ for each $a \in A$. Let

$$F := \mathrm{CH}\, g(A)$$

denote the set of feasible payoff vectors.[2]

There are three publicly observable outcomes: neither player transfers a good (outcome 0), player 1 transfers a good (outcome 1), and player 2 transfers a good (outcome 2). Let Y denote the set of all publicly observable outcomes, that is,

$$Y := \{0, 1, 2\}.$$

For each $a \in A$ and $y \in Y$, let $\pi(y \mid a)$ denote the probability with which y occurs when the players follow a stage-game strategy profile a:

$$\begin{aligned}
\pi(0 \mid a) &= p(1 - a_1) + p(1 - a_2) + (1 - 2p) \\
&= 1 - pa_1 - pa_2, \\
\pi(1 \mid a) &= pa_1, \\
\pi(2 \mid a) &= pa_2.
\end{aligned}$$

[2]We let $\mathrm{CH}\, S$ refer to the convex hull of a set $S \subset \mathbb{R}^N$, where $N \in \mathbb{N}$.

Note that outcome 0 occurs if either player 1 gets a good and consume it, player 2 gets a good and consume it, or neither player gets a good.

The game begins at period $t = 1$ and continues in an infinite horizon. A public history at the beginning of period t is a sequence $h^t = (y^1, \dots, y^{t-1})$. Let $H^1 := \{\emptyset\}$. The set of public histories at the beginning of period $t \geq 2$ is $H^t = Y^{t-1}$. Let $H := \bigcup_{t\geq 1} H^t$ denote the set of all public histories. A private history for player i at the beginning of period t is a sequence $h_i^t = (a_i^1, z_i^1, y^1, \dots, a_i^{t-1}, z_i^{t-1}, y^{t-1})$. Similarly, define $H_i^1 := \{\emptyset\}$, $H_i^t := C_i^{t-1}$, and $H_i := \bigcup_{t\geq 1} H_i^t$. A (repeated-game) strategy for player i is a map $\sigma_i \colon H_i \to \Delta(A_i)$. A strategy profile yields a probability distribution over histories in the obvious way. Each player tries to maximize his expected sum of the discounted stage game payoffs normalized by the factor of $1 - \delta$, where $\delta \in [0, 1)$ is the common discount factor. The payoff of player i when the players follow a repeated game strategy profile σ is defined by

$$E_\sigma \left[\sum_{t=1}^{\infty} (1-\delta)\delta^{t-1} g_i(a^t) \right],$$

where E_σ represents the expectation calculated from a strategy profile σ. A *public strategy* is a (repeated game) strategy that depends only on public history. That is, a public strategy is a map $\sigma_i \colon H \to \Delta(A_i)$. A *perfect public equilibrium* is a profile of public strategies such that, given any period t and public history h^t, the strategy profile is a Nash equilibrium from that period on. In this paper, I will focus on perfect public equilibria.

For each $\delta \in [0, 1)$, let $E^{\mathrm{f}}(\delta)$ be the (compact) set of perfect public equilibrium payoff vectors, and $W^*(\delta)$ be the maximum average payoff achieved by perfect public equilibria when the common discount factor is δ. That is,

$$W^*(\delta) := \max_{(u,v)\in E^{\mathrm{f}}(\delta)} \left[\frac{1}{2}u + \frac{1}{2}v \right].$$

By definition, $\{W^*(\delta) \mid \delta < 1\}$ is bounded above by the *efficient (average, ex ante) payoff* $\overline{W} := p\gamma$.

2.2 Chip Strategies

I study the performance of the chip strategy, which was first introduced by Möbius (2001) and further studied by Olszewski and Safronov (2018). Let n be a positive integer. Then, the *n-chip strategies* are defined as follows. At the beginning of each period, player $i \in \{1, 2\}$ (implicitly) holds k_i chips, where $k_i \in \{0, \ldots, 2n\}$ and $k_1 + k_2 = 2n$. If player i gets a good and $k_1 < 2n$, then he gives the good to his partner $j \neq i$ and the partner j gives a chip to player i in return. If the player does not get a good or if he gets a good but $k_i = 2n$, then he consumes the good himself and no chip is transferred. At the initial period, each player holds n chips. When a pair of n-chip strategies constitutes a perfect public equilibrium, I call this equilibrium an *n-chip-strategy equilibrium*.

For $\delta < 1$, $n \in \mathbb{N}$, and $k \in \{0, \ldots, 2n\}$, let $V_k^n(\delta)$ denote the value of holding k chips when the players follow the n-chip strategies. Since each player holds n chips at the initial period, the average payoff, denoted by $W_{\mathrm{c}}^n(\delta)$, can be expressed as

$$W_{\mathrm{c}}^n(\delta) = \frac{1}{2}V_n^n(\delta) + \frac{1}{2}V_n^n(\delta) = V_n^n(\delta).$$

For each $\delta < 1$, let $W_{\mathrm{c}}^*(\delta)$ denote the maximum average payoff achieved by the chip-strategy equilibria. That is,

$$W_{\mathrm{c}}^*(\delta) = \max_n W_{\mathrm{c}}^n(\delta),$$

where the maximum is taken over all n-chip-strategy equilibria when the discount factor is δ.[3,4] By definition, $W_{\mathrm{c}}^*(\delta) \leq W^*(\delta)$ for all $\delta < 1$.

Given a fixed discount factor $\delta < 1$, there exists a tradeoff between *efficiency* and *incentives* for the chip strategies; the average payoff $W_{\mathrm{c}}^n(\delta)$ is increasing in the number n of chips (whether or not the chip strategies constitute an equilibrium), while the incentive constraint becomes tighter as n increases. To see the efficiency part, consider a situation where players play the chip strategies with $2n$ total chips. Each player tries to exchange favors when his partner holds at least one chip, while he consumes the good himself (or equivalently punishes the partner) when he holds all the chips. Thus, the stage game average payoffs are $p\gamma = \overline{W}$ and $p(\gamma+1)/2 < \overline{W}$ for $(k_1, k_2) \neq (0, 2n), (2n, 0)$ and for $(k_1, k_2) = (0, 2n), (2n, 0)$, respectively. As the number n of chips increases, the time that players spends in $(0, 2n)$ or $(2n, 0)$ decreases, and hence the average payoff $W_{\mathrm{c}}^n(\delta)$ increases and approaches to $\overline{W}$.[5]

Next, let us consider the incentive constraints for the n-chip strategies to constitute an equilibrium. It is sufficient to check, by the one-shot deviation principle, whether

[3]When no chip-strategy equilibrium is sustained at $\delta < 1$, let $W_{\mathrm{c}}^*(\delta) = p$, which is the average payoff when each player consumes a good whenever possible.

[4]As subsequently discussed in this section, for each $\delta < 1$, there is an upper bound on the number n of chips such that n-chip strategies constitute an equilibrium, and hence the maximum is well defined.

[5]Formally, for each $\delta < 1$ the average payoff $W_{\mathrm{c}}^n(\delta)$ is (strictly) increasing in the number n of chips and $\lim_{n\to\infty} W_{\mathrm{c}}^n(\delta) = \overline{W}$, which follows from the closed form expression of $W_{\mathrm{c}}^n(\delta)$ in (20), Appendix C.1.

there is a profitable deviation from the prescribed (stage game) strategy. For $k = 2n$, a deviation is obviously not profitable, since no chip is transferred for any action. The incentive constraint for a player holding $k \in \{0, \dots, 2n - 1\}$ chips is as follows:

$$\delta V_{k+1}^n(\delta) \geq (1 - \delta) + \delta V_k^n(\delta), \tag{1}$$

where the left and the right-hand sides represent the payoffs from giving a good to his partner and from consuming a good himself, respectively, conditional on the event that he gets a good. Condition (1) gives an upper bound on the number n of chips such that the n-chip strategies constitute an equilibrium. To see this, consider the incentive for a player with $k = 2n - 1$. If he consumes a good, then he obtains a payoff of 1 and continues to hold $2n - 1$ chips. If he gives the good to his partner, then he gets a payoff of 0 and will hold $2n$ chips in the next period. Since punishment is applied when $k = 0$, a player is likely to be punished earlier if he deviates from the chip strategy. However, as n increases, the loss from the future punishment vanishes and is dominated by the gain from the deviation. Therefore, the incentive constraint cannot be sustained for large n. For each $\delta < 1$, let $n^*(\delta)$ denote the maximum number n of chips such that the n-chip strategies constitute an equilibrium when the discount factor is δ.

3 Results

There are some facts known about $W^*(\delta)$ and $W_c^*(\delta)$. The first is that the optimal perfect public equilibrium asymptotically achieves the efficient payoff. That is,

$$\lim_{\delta \to 1} W^*(\delta) = \overline{W}.$$

This follows from Fudenberg et al. (1994); for example, the Nash-threat folk theorem (Fudenberg et al. (1994, Theorem 6.1)) applies, since the action profile $a = (1, 1)$ has pairwise full rank.

Second, Olszewski and Safronov (2018, Proposition 1) show that the chip-strategy equilibrium asymptotically achieves the efficient payoff;

$$\lim_{\delta \to 1} W_c^*(\delta) = \overline{W}.$$

Third, the optimal perfect public equilibrium does not exactly achieve the efficient payoff for any fixed discount factor $\delta < 1$ (Proposition A.1 in Appendix A.1). Thus,

$$0 < \overline{W} - W^*(\delta) \leq \overline{W} - W^*_{\text{c}}(\delta).$$

Abdulkadiroglu and Bagwell (2012) show that for some set of parameter values, a "more sophisticated" favor-exchange equilibrium Pareto-dominates the chip-strategy equilibrium, and hence the weak inequality in fact holds with a strict inequality.

I show that the efficiency loss $\overline{W} - W^*_{\text{c}}(\delta)$ vanishes at the same rate as $\overline{W} - W^*(\delta)$, and the common rate of convergence is of order $(1-\delta)^{1/2}$.

Theorem 1.

$$\overline{W} - W^*_{\text{c}}(\delta) = \Theta\left((1-\delta)^{1/2}\right) \quad \textit{as} \quad \delta \to 1,$$

and

$$\overline{W} - W^*(\delta) = \Theta\left((1-\delta)^{1/2}\right) \quad \textit{as} \quad \delta \to 1.$$

In other words, the chip-strategy equilibrium achieves the efficient payoff $\overline{W}$ at the optimal rate, which is of order $(1-\delta)^{1/2}$. In particular, regarding the result of Abdulkadiroglu and Bagwell (2012), Theorem 1 implies that their "more sophisticated" strategies do not improve the rate of convergence for the efficiency loss.

I prove Theorem 1 by showing that (i) $\overline{W} - W^*(\delta)$ vanishes at rate at least $(1-\delta)^{1/2}$, and (ii) $\overline{W} - W^*_{\text{c}}(\delta)$ vanishes at rate at most $(1-\delta)^{1/2}$. Formally, I prove the following two propositions:

Proposition 1.

$$\overline{W} - W^*(\delta) = \Omega\left((1-\delta)^{1/2}\right) \quad \textit{as} \quad \delta \to 1.$$

Proposition 2.

$$\overline{W} - W^*_{\text{c}}(\delta) = O\left((1-\delta)^{1/2}\right) \quad \textit{as} \quad \delta \to 1.$$

Propositions 1 and 2 say that there exist $\kappa_1, \kappa_2 > 0$ and $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ we have $\overline{W} - W^*(\delta) > \kappa_1(1-\delta)^{1/2}$ and $\overline{W} - W^*_{\text{c}}(\delta) < \kappa_2(1-\delta)^{1/2}$. Since the inequality $\overline{W} - W^*(\delta) \leq \overline{W} - W^*_{\text{c}}(\delta)$ always holds, we have

$$\kappa_1(1-\delta)^{1/2} \leq \overline{W} - W^*(\delta) \leq \overline{W} - W^*_{\text{c}}(\delta) \leq \kappa_2(1-\delta)^{1/2}$$

for all $\delta \in (\underline{\delta}, 1)$, which implies Theorem 1. In the remainder of this section, I will make remarks on Propositions 1 and 2. The proofs are provided in the Appendix.

To prove Proposition 1, I show in Theorem A.1, Section A, that for any game satisfying a weak form of the full support condition, an extremal payoff vector (not necessarily efficient) is reached by equilibria at rate at least $(1-\delta)^{1/2}$. Formally, Theorem A.1 asserts that in any repeated game under imperfect public monitoring, if an extremal payoff vector is not supported in a stage-game Nash equilibrium and the corresponding pure action profile induces a full support distribution over public signals, then the distance between the payoff vector and the set of perfect public equilibrium payoff vectors is of order at least $(1-\delta)^{1/2}$. The favor-exchange game in the current paper satisfies the assumptions in Theorem A.1; the extremal payoff vector $(\overline{W}, \overline{W}) = g(1,1)$ is not supported in a stage-game Nash equilibrium, since each player i has a dominant strategy $a_i = 0$ in the stage game, and the corresponding action profile $(a_1, a_2) = (1,1)$ induces a full support distribution, that is, $\operatorname{supp} \pi(\cdot \mid 1,1) = \{0,1,2\} = Y$. Proposition 1 follows from Theorem A.1 (more precisely Corollary A.2), since the efficiency loss $\overline{W} - W^*(\delta)$ is of the same order as the distance between the extremal vector $(\overline{W}, \overline{W})$ and the equilibrium payoff set $E^{\mathrm{f}}(\delta)$ (Lemma B.1).

Theorem A.1 generalizes Proposition 4 of Hörner and Takahashi (2016) to accommodate the favor-exchange game, and in fact its proof utilizes the idea of their proof. Their Proposition 4 shows that in a repeated prisoner's dilemma under imperfect public monitoring with full support, the distance between the set of perfect public equilibrium payoff vectors and the full cooperation payoff vector vanishes at rate at least $(1-\delta)^{1/2}$. Note that Theorem A.1 in the current paper applies to any repeated game beyond the prisoner's dilemma, and to any monitoring structure such that the pure action profiles associated with the targeted payoff vector induce full support distributions, while Proposition 4 in Hörner and Takahashi (2016) requires all pure action profiles to induce full support distributions. In fact, the favor-exchange game does not satisfy the full support condition of their Proposition 4, since for each player i, the outcome $y = i$, which means that player i transfers a good to his partner, will never realize if he chooses the (stage-game) strategy $a_i = 0$. On the other hand, as already stated above, the favor-exchange game does satisfy the weakened condition of Theorem A.1.

Proposition 2 is proved in three steps. First, I derive a closed form expression of $V_k^n(\delta)$, the value of holding k chips when players follow the n-chip strategies (Section C.1). This provides a closed form expression of the efficiency loss $\overline{W} - W_{\mathrm{c}}^n(\delta)$. Second, with the closed form expression of $V_k^n(\delta)$, I derive the divergence rate of $n^*(\delta)$, the maximum number n of chips which can be supported in chip-strategy equilibria when the discount factor is δ. Formally, I show in Lemma C.4 that

$$n^*(\delta) = \Omega\left(\frac{1}{(1-\delta)^{1/2}}\right).$$

Third, I derive the convergence rate of $\overline{W} - W_c^*(\delta)$ by using the closed form expression of the efficiency loss $\overline{W} - W_c^n(\delta)$ and the divergence rate of $n^*(\delta)$.

The above analysis refines that of Olszewski and Safronov (2018) used for their asymptotic efficiency result. They show that for any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$, the n-chip strategies constitute an equilibrium and satisfy $W_c^n(\delta) > \overline{W} - \varepsilon$. In their Lemma 2 for the above result, they prove that for any number n of chips there exists $\underline{\delta}_n < 1$ such that for every $\delta > \underline{\delta}_n$, the n-chip strategies constitute an equilibrium, which in particular implies that $n^*(\delta) \to \infty$ as $\delta \to 1$. In the proof of Lemma 2, they obtain a first-order approximation of the values $V_k^n(\delta)$ near $\delta = 1$ by applying the implicit function theorem to the recursive equation of the values $V_k^n(\delta)$ (Equation (13) in Section C.1 in the present paper). In the present paper, instead, I derive a closed form expression of $V_k^n(\delta)$ by solving the recursive equation (13) explicitly. This enables me to obtain the divergence rate of $n^*(\delta)$, and in turn the convergence rate of $\overline{W} - W_c^*(\delta)$.

My analysis also enables me to identify the leading coefficients for $n^*(\delta)$ and $\overline{W} - W_c^*(\delta)$. Formally, $n^*(\delta)$ and $\overline{W} - W_c^*(\delta)$ are in fact written as

$$n^*(\delta) = c_{p,\gamma}(1-\delta)^{-1/2} + o\big((1-\delta)^{-1/2}\big),$$

and

$$\overline{W} - W_c^*(\delta) = C_{p,\gamma}(1-\delta)^{1/2} + o\big((1-\delta)^{1/2}\big),$$

where I derive the leading coefficients $c_{p,\gamma}$ and $C_{p,\gamma}$ explicitly (Proposition D.1).

4 Conclusion

I study the performance of the chip-strategy equilibrium, relative to the optimal perfect public equilibrium, in terms of the rate at which these equilibria achieve the efficient payoff as the discount factor δ tends to 1. The main result of this paper is that the chip-strategy equilibrium is optimal in the convergence rate, that is, the chip-strategy equilibrium achieves the efficient payoff at the same rate as the optimal equilibrium. Formally, I show in Proposition 1 that $\overline{W} - W^*(\delta)$, the efficiency loss for the optimal equilibrium, vanishes at rate at least $(1-\delta)^{1/2}$, and in Proposition 2 that $\overline{W} - W_c^*(\delta)$, the efficiency loss for the chip-strategy equilibrium, vanishes at rate at most $(1-\delta)^{1/2}$.

The main results, in fact, extend, with some modification, to the asymmetric case in which a favor opportunity arrives with unequal probabilities for two players. For this case, Olszewski and Safronov (2018) show that the chip strategies are not asymptotically efficient, that is, the efficiency loss for the chip strategies is uniformly bounded away from 0 for any $\delta < 1$, and that asymptotic efficiency is achieved by random chip strategies where players decide whether or not to give a chip in return for a favor depending on public randomization outcomes. An appropriate modification of the proof of Proposition 2 in the present paper shows that the random-chip-strategy equilibrium also achieves efficiency at rate at most $(1 - \delta)^{1/2}$. For Proposition 1 and Theorem A.1, I have to modify the proof of Theorem A.1 to accommodate perfect public equilibria with public randomization, by which it is shown that the optimal equilibrium still achieves efficiency at rate at least $(1 - \delta)^{1/2}$.[6] Hence, public randomization does not improve the rate of convergence.

A possible future research direction is investigating whether or not the results of this paper still hold in the class of games considered by Olszewski and Safronov (2018), in which the chip-strategy equilibria are shown to be asymptotically efficient. In fact, all the results in the present paper should hold in this class if we assume that types are i.i.d. over time (but may be correlated across players) instead of following a Markov process; the techniques used in the proofs of Propositions 1, 2, and Theorem A.1 should be applicable to the class of games. The remaining work is to explore the case in which types follow a Markov process.

A Lower Bound on the Rate of Convergence for General Games

A.1 General Model of Repeated Games under Imperfect Public Monitoring

In this section, I will formally state and prove Theorem A.1. To this end, I will introduce a general model of repeated games under imperfect public monitoring.

There are $N \geq 2$ players. Each player i has a nonempty finite set A_i of actions, and a stage game payoff function $g_i \colon A \to \mathbb{R}$, where $A := \times_i A_i$. Define $g \colon A \to \mathbb{R}^N$ by $g(a) = (g_1(a), \ldots, g_N(a))$ for each $a \in A$. Let $F := \mathrm{CH}\, g(A)$ denote the set of feasible payoff vectors. Monitoring is imperfect and public. Let Y be a nonempty and finite set of public signals. For each $a \in A$ and $y \in Y$, let $\pi(y \mid a)$ denote the probability with which the public signal y realizes when players follow the action profile a. For each probability distribution $\alpha \in \Delta(A)$, define $g(\alpha)$ and $\pi(\cdot \mid \alpha)$ as usual. For each

[6] The modified proofs are available upon request.

mixed action profile $\alpha \in \times_i \Delta(A_i)$, with a slight abuse of notation, denote again by α the resulted probability distribution over A.

The game begins at period $t = 1$ and continues in an infinite horizon. A public history at the beginning of period t is a sequence $h^t = (y^1, \ldots, y^{t-1})$. Let $H^1 := \{\emptyset\}$. The set of public histories at the beginning of period $t \geq 2$ is $H^t := Y^{t-1}$. Let $H := \bigcup_{t \geq 1} H^t$ denote the set of all public histories.

A private history for player i at the beginning of period t is a sequence $h_i^t = (a_i^1, y^1, \ldots, a_i^{t-1}, y^{t-1})$. Similarly, define $H_i^1 := \{\emptyset\}$, $H_i^t := (A_i \times Y)^{t-1}$, and $H_i := \bigcup_{t \geq 1} H_i^t$. A (repeated-game) strategy for player i is a map $\sigma_i \colon H_i \to \Delta(A_i)$. A strategy profile yields a probability distribution over histories in the obvious way. The payoff of player i when players follow a repeated game strategy profile σ is defined by

$$E_\sigma \left[\sum_{t=1}^{\infty} (1 - \delta) \delta^{t-1} g_i(a^t) \right],$$

where E_σ represents the expectation calculated from a strategy profile σ. A *public strategy* is a (repeated game) strategy that depends only on public history. That is, a public strategy is a map $\sigma_i \colon H \to \Delta(A_i)$. A *perfect public equilibrium* is a profile of public strategies such that, given any period t and public history h^t, the strategy profile is a Nash equilibrium from that period on. For each $\delta \in [0, 1)$, let $E(\delta)$ be the set of payoff vectors supported in perfect public equilibria.

The following proposition provides a sufficient condition for a payoff vector not to be supported in any perfect public equilibrium:

Proposition A.1 (A modification of Theorem 6.5 of Fudenberg et al. (1994))**.** *Let $\bar{v}$ be an extremal payoff vector of F. If for each mixed action profile $\alpha \in \times_i \Delta(A_i)$ with $g(\alpha) = \bar{v}$ there exist a player j and an action $a_j' \in A_j$ such that (i) $g_j(a_j', \alpha_{-j}) > g_j(\alpha)$ and (ii) $\operatorname{supp} \pi(\cdot \mid a_j', \alpha_{-j}) \subset \operatorname{supp} \pi(\cdot \mid \alpha)$, then*

$$\bar{v} \notin E(\delta) \tag{2}$$

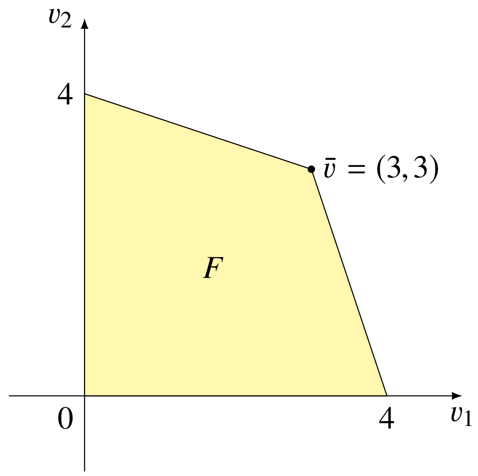
for each $\delta < 1$.[7]

Proposition A.1 modifies Theorem 6.5 of Fudenberg et al. (1994). They incorrectly assert that for an extremal payoff vector $\bar{v} \in F$, if for each pure action profile $a \in A$ with $g(a) = \bar{v}$ there exist a player j and an action $a_j' \in A_j$ such that (i) $g_j(a_j', a_{-j}) > g_j(a)$ and (ii) $\operatorname{supp} \pi(\cdot \mid a_j', a_{-j}) \subset \operatorname{supp} \pi(\cdot \mid a)$, then $\bar{v} \notin E(\delta)$ for all $\delta < 1$. Their original

[7]For each probability distribution π over a finite set, denote its support by $\operatorname{supp} \pi$.

	LL	LR	RL	RR
TT	3, 3	3, 3	0, 4	0, 0
TB	3, 3	3, 3	0, 0	0, 4
BT	4, 0	0, 0	0, 0	0, 0
BB	0, 0	4, 0	0, 0	0, 0

(a) The payoff matrix.



(b) The set of feasible payoff vectors.

Fig. 1: A counter example for Theorem 6.5 in Fudenberg et al. (1994).

assumptions do not exclude the possibility that an extremal payoff vector $\bar{v}$ is supported in a stage-game Nash equilibrium and hence in a perfect public equilibrium. Consider the stage game in Fig. 1 under imperfect public monitoring with full support, that is, $\pi(\cdot \mid a) = Y$ for all $a \in A$. The extremal payoff vector $\bar{v} = (3, 3)$ satisfies the original assumptions of Fudenberg et al. (1994, Theorem 6.5); for each pure action profile $a \in A$ with $g(a) = \bar{v} = (3, 3)$, there exist a player j and an action $a'_j \in A_j$ such that $g_j(a'_j, a_{-j}) > g_j(a)$ and $\operatorname{supp} \pi(\cdot \mid a'_j, a_{-j}) \subset \operatorname{supp} \pi(\cdot \mid a)$. However, it is supported in a stage-game Nash equilibrium $((1/2)TT + (1/2)TB, (1/2)LL + (1/2)LR)$, and hence is a perfect public equilibrium payoff vector for all $\delta < 1$.

Note that if an extremal payoff vector $\bar{v}$ is uniquely achieved by a pure action profile a, then the assumptions of Proposition A.1 coincide with the original ones of Theorem 6.5 in Fudenberg et al. (1994). One can prove Proposition A.1 by simply replacing the pure action profile a by the mixed action profile α in the original proof of Fudenberg et al. (1994, Theorem 6.5).

A.2 Results

Theorem A.1. *Let $\bar{v}$ be an extremal payoff vector of F that is not supported in any stage-game Nash equilibrium. If each pure action profile $a \in A$ with $g(a) = \bar{v}$ induces a full support distribution, that is,* $\operatorname{supp} \pi(\cdot \mid a) = Y$ *for each $a \in A$ with $g(a) = \bar{v}$,* *then*

$$d(\bar{v}, E(\delta)) = \Omega\left((1-\delta)^{1/2}\right) \quad as \quad \delta \to 1.$$[8]

Theorem A.1 gives a *tight* lower bound on rate of convergence. For example, in my favor-exchange game, the distance between $E(\delta)$ and $(\overline{W}, \overline{W})$ vanishes at rate exactly $(1-\delta)^{1/2}$. Moreover, in "most" situations, the distance between the equilibrium payoff set and an extremal payoff vector vanishes at rate exactly $(1-\delta)^{1/2}$. In fact, Hörner and Takahashi (2016) show in their Proposition 8 that the distance vanishes at rate at most $(1-\delta)^{1/2}$ if (i) the targetted payoff vector is strictly individually rational, (ii) the monitoring structure satisfies a certain condition sufficient for the folk theorem, and (iii) the payoff function satisfies a mild genericity condition.

The full support condition in Theorem A.1 is indispensable. Consider a game under perfect monitoring, in which each strictly individually rational payoff vector is exactly achieved in equilibria (i.e., the rate of convergence is 0). Another example is a game with the all-or-nothing monitoring structure, where players randomly observe the chosen action profile or nothing. The rate of convergence is also 0 under this monitoring structure. This follows from Lemma 7 of Hörner and Takahashi (2016), which shows that the rate of convergence for the equilibrium payoff sets is the same under the all-or-nothing monitoring structure as under perfect monitoring.

The assumptions in Theorem A.1 are stronger than those in Proposition A.1. Note that if an extremal payoff vector $\bar{v}$ satisfies the full support condition in Theorem A.1, then condition (ii) in Proposition A.1 holds for each i, $a_i' \in A_i$, and each $\alpha \in \times_i \Delta(A_i)$ with $g(\alpha) = \bar{v}$. Hence if, in addition, $\bar{v}$ is not supported in any stage-game Nash equilibrium, then it satisfies the assumptions in Proposition A.1. It is an open question whether $d(\bar{v}, E(\delta)) = \Omega((1-\delta)^{1/2})$, the conclusion of Theorem A.1, still holds under the assumptions in Proposition A.1. (A game with the all-or-nothing monitoring structure above does not satisfy the assumptions in Proposition A.1.)

Theorem A.1 can be rephrased as follows:

Corollary A.1. *Let $\bar{v}$ be an extremal payoff vector of F. If* $\operatorname{supp} \pi(\cdot \mid a) = Y$ *for each $a \in A$ with $g(a) = \bar{v}$, then the following statements are equivalent:*

(i) $\bar{v}$ is not supported in any stage-game Nash equilibrium.

(ii) $\bar{v}$ is not supported in any perfect public equilibrium for any discount factor $\delta < 1$, that is, $\bar{v} \notin E(\delta)$ for each $\delta < 1$.

---

[8]For each $v = (v_1, \ldots, v_n) \in \mathbb{R}^N$, let $\|v\| := \sqrt{\sum_i v_i^2}$. For each $v \in \mathbb{R}^N$ and each $S \subset \mathbb{R}^N$ with $S \neq \emptyset$, let $d(v, S) := \inf_{v' \in S} \|v - v'\|$.

(iii) $d(\bar{v}, E(\delta)) = \Omega\big((1-\delta)^{1/2}\big)$ *as* $\delta \to 1$.

I apply Theorem A.1 to the favor-exchange game in the form of Corollary A.2 below. Note the difference between $E^{\mathrm{f}}(\delta)$ and $E(\delta)$ in definition; the former is the set of perfect public equilibrium payoffs for which players are allowed to deviate to a strategy as a mapping from $\bigcup_{t\geq 1}(A_i \times Z_i \times Y)^{t-1}$ to $\Delta(A_i)$, while the latter is the set of perfect public equilibrium payoffs for which players are allowed to deviate to a strategy as a mapping from $\bigcup_{t\geq 1}(A_i \times Y)^{t-1}$ to $\Delta(A_i)$. Nevertheless, $E^{\mathrm{f}}(\delta) = E(\delta)$ for each $\delta < 1$. By definition, $E^{\mathrm{f}}(\delta) \subset E(\delta)$. The opposite inclusion $E^{\mathrm{f}}(\delta) \supset E(\delta)$ also holds, since no player can be better off by utilizing private information given that his opponents follow public strategies.

Corollary A.2. *Consider the favor-exchange game described in Section 2.1. Let* $\bar{v} := (\overline{W}, \overline{W})$. *We have*

$$d\big(\bar{v}, E^{\mathrm{f}}(\delta)\big) = \Omega\big((1-\delta)^{1/2}\big) \quad \textit{as} \quad \delta \to 1. \tag{3}$$

Proof. The extremal payoff vector $\bar{v} = g(1,1)$ is not supported in any stage-game Nash equilibrium, since each player i has a dominant strategy $a_i = 0$ in the stage game. Moreover, the corresponding pure action profile $(a_1, a_2) = (1,1)$ induces a full support distribution; $\pi(\cdot \mid 1,1) = \{0,1,2\} = Y$. Thus Theorem A.1 applies, that is,

$$d\big(\bar{v}, E(\delta)\big) = \Omega\big((1-\delta)^{1/2}\big) \quad \text{as} \quad \delta \to 1 \tag{4}$$

Since $E(\delta) = E^{\mathrm{f}}(\delta)$ for each $\delta < 1$, we obtain (3) from (4). □

A.3 Proof of Theorem A.1

I prove Theorem A.1 by modifying the technique used in Proposition 4 of Hörner and Takahashi (2016). Note that while their proof idea is applicable in more general settings, their proof is simplified due to the following special features of the game that they consider: (i) the full support condition on the monitoring structure, (ii) the existence of a stage-game dominant strategy. (See footnotes 14 and 15.) To prove Theorem A.1 without relying on these features (in particular (ii)), I prepare the following technical lemma:

Lemma A.1. *Let B be an arbitrary nonempty subset of A. For any $\bar{\xi} > 0$, there exists $\chi > 0$ such that for all $\alpha \in \times_i \Delta(A_i)$ with $\sum_{a\in B} \alpha(a) > 1 - \chi$, there exist $\xi \in [0, \bar{\xi})$, $\bar{\alpha} \in \times_i \Delta(A_i)$, and $\underline{\alpha} \in \Delta(A)$ such that*

- $\alpha = (1-\xi)\bar{\alpha} + \xi\underline{\alpha}$; *and*

- $\operatorname{supp}\bar{\alpha} \subset B$.

In words, Lemma A.1 asserts that a mixed action profile which is highly concentrated on a subset B can be approximated by a mixed action profile the support of which is contained in B.

Proof of Lemma A.1. Let $\mathcal{A} = \{A'_1 \times \cdots \times A'_N \mid A'_i \subset A_i \text{ for all } i = 1, \ldots, N\}$. Fix any nonempty set $B \subset A$.

First, I show that for any $\bar{\xi} > 0$, there exists $\chi > 0$ such that for all $\alpha \in \times_i \Delta(A_i)$ with $\sum_{a \in B} \alpha(a) > 1 - \chi$, there exists a set $A' \subset B$ such that $A' \in \mathcal{A}$ and $\sum_{a \in A'} \alpha(a) > 1 - \bar{\xi}$. Suppose for a contradiction that there exists $\bar{\xi} > 0$ such that for any $\chi > 0$ there exists $\alpha \in \times_i \Delta(A_i)$ such that $\sum_{a \in B} \alpha(a) > 1 - \chi$ and $\sum_{a \in A'} \alpha(a) \leq 1 - \bar{\xi}$ for all $A' \subset B$ with $A' \in \mathcal{A}$. For each $m \in \mathbb{N}$, let $\alpha^m \in \times_i \Delta(A_i)$ be such that $\sum_{a \in B} \alpha^m(a) > 1 - 1/m$ and $\sum_{a \in A'} \alpha^m(a) \leq 1 - \bar{\xi}$ for all $A' \subset B$ with $A' \in \mathcal{A}$. Since $\times_i \Delta(A_i)$ is a compact set, there exist a subsequence $\{\alpha^{m(k)}\}$ of $\{\alpha^m\}$ and $\bar{\alpha} \in \times_i \Delta(A_i)$ such that $\alpha^{m(k)} \to \bar{\alpha}$ as $k \to \infty$. By construction, we have $\sum_{a \in B} \bar{\alpha}(a) = 1$ and $\sum_{a \in A'} \bar{\alpha}(a) \leq 1 - \bar{\xi}$ for all $A' \subset B$ with $A' \in \mathcal{A}$. Let $A' = \operatorname{supp}\bar{\alpha}$. By definition, $\sum_{a \in A'} \bar{\alpha}(a) = 1$. Since $\bar{\alpha} \in \times_i \Delta(A_i)$, we must have $A' = \operatorname{supp}\bar{\alpha} \in \mathcal{A}$. Moreover, we have $A' \subset B$, since $\sum_{a \in B} \bar{\alpha}(a) = 1$. However, these contradict the fact that $\sum_{a \in A'} \bar{\alpha}(a) \leq 1 - \bar{\xi}$.

Now, take any $\bar{\xi} \in (0, 1)$. As I have proved above, we can find $\chi > 0$ such that for all $\alpha \in \times_i \Delta(A_i)$ with $\sum_{a \in B} \alpha(a) > 1 - \chi$, there exists a set $A' \subset B$ such that $A' \in \mathcal{A}$ and $\sum_{a \in A'} \alpha(a) > 1 - \bar{\xi}$. Take any $\alpha \in \times_i \Delta(A_i)$ with $\sum_{a \in B} \alpha(a) > 1 - \chi$, and let $A' \subset B$ be such that $A' \in \mathcal{A}$ and $\sum_{a \in A'} \alpha(a) > 1 - \bar{\xi}$. Let $\xi = \sum_{a \notin A'} \alpha(a) \, (< \bar{\xi})$. For each i, define $\bar{\alpha}_i \colon A_i \to \mathbb{R}$ by

$$\bar{\alpha}(a_i) = \frac{\alpha_i(a_i)\mathbb{1}\{a_i \in A'_i\}}{\sum_{a'_i \in A'_i} \alpha_i(a'_i)}$$

for each $a_i \in A_i$, and let $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$.[9] Note that $\bar{\alpha}_i \in \Delta(A_i)$ for each i, and

$$\bar{\alpha}(a) = \frac{\alpha(a)\mathbb{1}\{a \in A'\}}{\sum_{a' \in A'} \alpha(a')}$$

for all $a \in A$. Further, define $\underline{\alpha} \colon A \to \mathbb{R}$ by

$$\underline{\alpha}(a) = \frac{\alpha(a)\mathbb{1}\{a \notin A'\}}{\sum_{a' \notin A'} \alpha(a')}$$

for each $a \in A$.[10] We obtain $\underline{\alpha} \in \Delta(A)$, and $\alpha = (1-\xi)\bar{\alpha} + \xi\underline{\alpha}$.

[9] $\mathbb{1}\{\cdot\}$ denotes the indicator that switches on if and only if the input statement is true.

[10] If $\sum_{a' \notin A'} \alpha(a) = 0$, then let $\underline{\alpha}$ be an arbitrary probability distribution over A.

□

Now, we will prove Theorem A.1. The proof contains several lemmas. To clarify the argument, I will relegate the proofs of the lemmas to the next subsection A.4.

Proof of Theorem A.1. Fix an extremal payoff vector $\bar{v} \in F$ that is not supported in any stage-game Nash equilibrium, and suppose that $\operatorname{supp} \pi(\cdot \mid a) = Y$ for all $a \in g^{-1}(\bar{v})$.[11] Let $L := \max_{a,a' \in A} \|g(a) - g(a')\|$ and $\eta := \min_{a \in g^{-1}(\bar{v})} \min_{y \in Y} \pi(y \mid a)$. Note that $\eta > 0$, since $\pi(y \mid a) > 0$ for all $y \in Y$ and $a \in g^{-1}(a)$. Let $\nu > 0$ be so that for any $\alpha \in \times_i \Delta(A_i)$ with $g(\alpha) = \bar{v}$ there exist a player i and $a_i' \in A_i$ such that $g_i(a_i', \alpha_{-i}) - g_i(\alpha) > \nu$.[12]

By Gordan's theorem, we can find a unit vector $\lambda_0 \in \mathbb{R}^N$ such that $\lambda_0 \cdot v' < \lambda_0 \cdot \bar{v}$ for all $v' \in F \setminus \{\bar{v}\}$. Fix $\theta \in (0,1)$ sufficiently close to 1.[13] For each $q > 0$, denote by B_q the closed ball with center $\bar{v} - q\lambda_0$ and radius $r = \theta q$, and by S_q its sphere/boundary. Note that for θ sufficiently close to 1, sphere S_q crosses each ray $\{\bar{v} + t(g(a) - \bar{v}) \mid t \geq 0\}$, where $a \notin g^{-1}(a)$.

Let F_q be the subset of F that excludes the neighborhood of $\bar{v}$ separated by the sphere S_q. For each $\delta < 1$, we can find the largest q such that $E(\delta) \subset F_q$, since $E(\delta)$ is a compact set and $\bar{v} \notin E(\delta)$ (Proposition A.1). Let $q(\delta) = q$, and $r(\delta) = \theta q(\delta)$. Since the distance between $E(\delta)$ and $\bar{v}$ is at least $(1 - \theta)q(\delta)$, it suffices to show that $q(\delta)$ is of order at least $(1 - \delta)^{1/2}$.

Now, fix any $\delta < 1$ sufficiently close to 1. By the compactness of $E(\delta)$ and the maximality of $q(\delta)$, we can find an equilibrium payoff vector $v \in E(\delta)$ at which $E(\delta)$ touches the sphere $S_{q(\delta)}$ (Fig. 2). Let λ denote the unit vector normal to the sphere $S_q(\delta)$ at point v;

$$\lambda = \frac{v - \bar{v} + q(\delta)\lambda_0}{\theta q(\delta)}.$$

The following lemmas are graphically evident.

Lemma A.2. *For each $v' \in F_{q(\delta)}$, we have*

$$\lambda \cdot v' \leq \lambda \cdot v.$$

Lemma A.3. *For each $v' \in F$, we have*

$$\lambda \cdot v' \leq \lambda \cdot \bar{v}.$$

[11] For each $v \in \mathbb{R}^N$, let $g^{-1}(v) := \{a \in A \mid g(a) = v\}$.

[12] Existence of such ν follows from the assumption that $\bar{v}$ is not supported by any stage-game Nash equilibrium, and the Extreme Value Theorem.

[13] The formal definition of θ is provided in Section A.4 to prove technical lemmas.

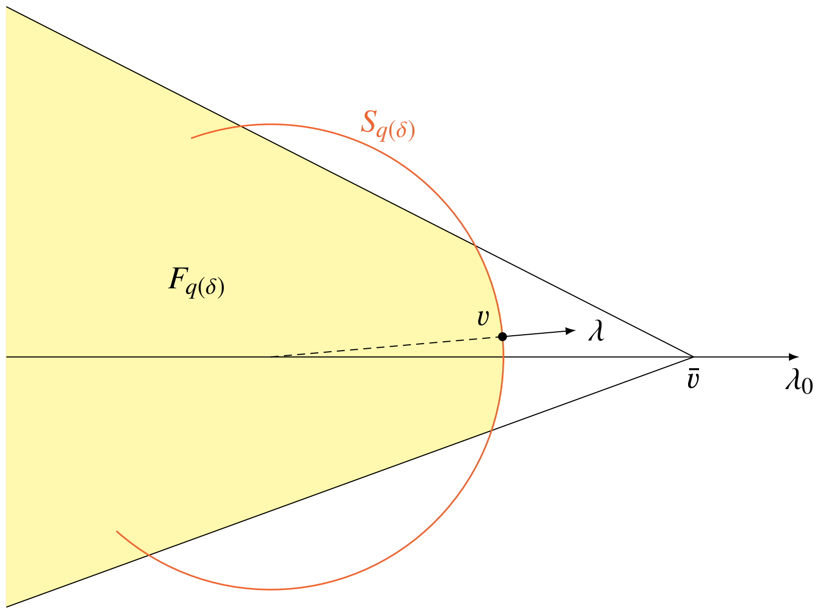


Fig. 2: The shaded area depicts $F_{q(\delta)}$, and the arc depicts a part of the sphere $S_{q(\delta)}$. The node on the arc represents an equilibrium payoff vector $v \in E(\delta)$ satisfying at which the equilibrium payoff set $E(\delta)$ touches $S_{q(\delta)}$. The unit vector λ is normal to $S_{q(\delta)}$.

Fix any perfect public equilibrium associated with payoff profile v. Let $\alpha \in \times_i \Delta(A_i)$ be the mixed action profile played in the initial period, and $w(y)$ be the continuation payoff vector after each public signal $y \in Y$. First, we will show that $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at least $1-\delta$. This is verified by the following simple argument if the mixed action profile α satisfies $g(\alpha) = \bar{v}$. To this end, suppose that $g(\alpha) = \bar{v}$. We can find a player i and an action $a_i' \in A_i$ such that $g_i(a_i', \alpha_{-i}) - g_i(\alpha) > \nu$. Since the equilibrium strategy of player i prescribes α_i, we must have

$$(1-\delta)g_i(\alpha) + \delta \sum_{y \in Y} \pi(y \mid \alpha) w(y) \geq (1-\delta) g_i(a_i', \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y \mid a_i', \alpha_{-i}) w(y).$$

Since $g_i(a_i', \alpha_{-i}) - g_i(\alpha) > \nu$, there exist $y, y' \in Y$ such that $w_i(y) - w_i(y') \geq \nu(1-\delta)/\delta$, which implies that $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at least $1-\delta$.

Now, we will remove the restriction that $g(\alpha) = \bar{v}$, by approximating α by some mixed action profile $\bar{\alpha}$ with $g(\bar{\alpha}) = \bar{v}$.

Lemma A.4. *For any $\bar{\xi} > 0$, there exists $\chi > 0$ such that if $q(\delta) < \chi$ then there exist $\xi \in [0, \bar{\xi})$, $\bar{\alpha} \in \times_i \Delta(A_i)$, and $\underline{\alpha} \in \Delta(A)$ such that $\alpha = (1-\xi)\bar{\alpha} + \xi \underline{\alpha}$ and $g(\bar{\alpha}) = \bar{v}$, where χ depends on $\bar{\xi}$ and the stage-game structure, but not on δ or the choice of v and α.*

For $\bar{\xi} = \nu/2(\nu + L)$, let $\chi > 0$ be as stated in the above lemma. We can assume $q(\delta) < \chi$, and then let $\bar{\xi}$, $\bar{\alpha} \in \Delta$, and $\underline{\alpha} \in \Delta(A)$ be such that $\alpha = (1-\xi)\bar{\alpha} + \xi\underline{\alpha}$ and $g(\bar{\alpha}) = \bar{v}$. Since $\bar{\alpha} \in \times_i \Delta(A_i)$ satisfies $g(\bar{\alpha}) = \bar{v}$, we can find i and $a_i' \in A_i$ such that $g_i(a_i', \bar{\alpha}_{-i}) - g_i(\bar{\alpha}) > \nu$. By optimality of α_i, we must have

$$(1-\delta)g_i(a_i', \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y \mid a_i', \alpha_{-i}) w_i(y) \le (1-\delta)g_i(\alpha) + \delta \sum_{y \in Y} \pi(y \mid \alpha) w_i(y),$$

or equivalently

$$\frac{1-\delta}{\delta}\left(g_i(a_i', \alpha_{-i}) - g_i(\alpha)\right) \le \sum_{y \in Y} \pi(y \mid \alpha) w_i(y) - \sum_{y \in Y} \pi(y \mid a_i', \alpha_{-i}) w_i(y). \tag{5}$$

Note that

$$\begin{aligned} g_i(a_i', \alpha_{-i}) - g_i(\alpha) &= \sum_{a \in A} \alpha(a)(g_i(a_i', a\_i) - g_i(a)) \\ &= (1-\xi)\sum_{a \in A} \bar{\alpha}(g_i(a_i', a_{-i}) - g_i(a)) + \xi \sum_{a \in A} \underline{\alpha}(a)(g_i(a_i', a_{-i}) - g_i(a)) \\ &\ge (1-\xi)(g_i(a_i', \bar{\alpha}_{-i}) - g_i(\bar{\alpha})) - \xi L \\ &\ge (1-\xi)\nu - \xi L \\ &> \frac{\nu}{2}. \end{aligned}$$

Therefore, there exist $y, y' \in Y$ such that

$$w_i(y) - w_i(y') \ge \nu(1-\delta)/2\delta, \tag{6}$$

and hence $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at least $1-\delta$.[14]

Second, we will show that $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at most $q(\delta)(1-\delta)^{1/2}$ by proving that (i) each continuation payoff vector $w(y)$ is distant from v in the direction of λ with order at most $q(\delta)(1-\delta)$ and (ii) all the continuation payoffs $w(y)$ lie in the closed ball $B_{q(\delta)}$ (Lemma A.5). To prove (i), take any $y \in Y$. Since $\lambda \cdot g(\alpha) \le \lambda \cdot \bar{v}$ (Lemma A.2) and $\lambda \cdot w(y') \le \lambda \cdot v$ for all $y' \in Y$ (Lemma A.3), we have

$$\lambda \cdot v = (1-\delta)\lambda \cdot g(\alpha) + \delta \sum_{y' \in Y} \pi(y' \mid \alpha) \lambda \cdot w(y')$$

[14]In the prisoner's dilemma analyzed by Hörner and Takahashi (2016), such an elaborate approximation is not necessary. We can find a positive constant $\nu' > 0$ such that $g_1(D, \alpha_2) - g_1(C, \alpha_2) > \nu'$ for any mixed action α_2 of player 2, where C and D are defined as usual. What remains is to show that α_1 puts a positive probability on C, which can be shown in a much easier way.

$$\leq (1-\delta)\lambda \cdot \bar{v} + \delta((1-\pi(y \mid \alpha))\lambda \cdot v + \pi(y \mid \alpha)\lambda \cdot w(y)).$$

Note that the probability with which each public signal realizes in the mixed action profile α is uniformly bounded away from zero;[15]

$$\begin{aligned} \pi(y \mid \alpha) &= (1-\xi)\pi(y \mid \bar{\alpha}) + \xi\pi(y \mid \underline{\alpha}) \\ &\geq (1-\xi)\eta \\ &\geq \eta/2. \end{aligned} \tag{7}$$

Therefore, we have

$$\lambda \cdot (w(y) - v) \geq -\varepsilon(\delta), \tag{8}$$

where

$$\varepsilon(\delta) := \frac{2}{\delta\eta}(1-\delta)\lambda \cdot (v - \bar{v}).$$

Since $\lambda \cdot (v - \bar{v})$ is of order $q(\delta)$ and $\lambda \cdot (w(y) - v) \leq 0$ (Lemma A.2), we have shown (i) that the continuation payoff $w(y)$ is distant from v in the direction of λ at most $\varepsilon(\delta)$, which is of order $q(\delta)(1-\delta)$.

Lemma A.5. *For $\delta < 1$ sufficiently close to 1, each continuation payoff vector $w(y)$ is inside the closed ball $B_{q(\delta)}$. That is,*

$$\|w(y) - \bar{v} + q(\delta)\lambda_0\| \leq r(\delta)$$

for all $y \in Y$.

Since each $w(y)$ is distant from v in the direction of λ at most $\varepsilon(\delta)$ and lies in the closed ball $B_{q(\delta)}$, the continuation payoff vectors $w(\cdot)$ can vary by at most $2(r(\delta)^2 - (r(\delta) - \varepsilon(\delta))^2)^{1/2}$, which is of order $q(\delta)(1-\delta)^{1/2}$. That is, $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at most $q(\delta)(1-\delta)^{1/2}$.

The above arguments have shown that $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at least $(1-\delta)$ and simultaneously of order at most $q(\delta)(1-\delta)^{1/2}$. Therefore, $q(\delta)$ must be of order at least $(1-\delta)^{1/2}$. □

The following two parts are keys for the proof of Theorem A.1 (as well as that of Hörner and Takahashi (2016, Proposition 4)):

(i) $\max_{y,y' \in Y} \|w(y) - w(y')\|$ is of order at least $1-\delta$ (inequality (6)).

[15] This is evident if the monitoring structure satisfies the full support condition, which is assumed in Hörner and Takahashi (2016, Proposition 4).

(ii) $\max_{y\in Y}|\lambda \cdot (w(y) - v)|$ is of order at most $q(\delta)(1-\delta)$ (inequality (8)).

Part (i) holds, since the difference between continuation payoffs $\sum_{y\in Y} \pi(y \mid \alpha)w_i(y)$ and $\sum_{y\in Y} \pi(y \mid a_i', \alpha_{-i})w_i(y)$ must be of order at least $1-\delta$, which is the order of the flow gain from deviation from α_i to a_i' (inequality (5)). Part (ii) follows from Lemmas A.2, A.3, and the full support assumption. In the proof of part (ii), I use the full support assumption to show the probabilities $\pi(y \mid \alpha)$ are uniformly bounded away from 0 (inequality (7)). Part (ii) then implies that $\max_{y,y'\in Y}\|w(y) - w(y')\|$ is of order at most $q(\delta)(1-\delta)^{1/2}$, since all continuation payoffs lie in the ball $B_{q(\delta)}$ with radius $r = \theta q(\delta)$.

A.4 Proofs of Lemmas

Let $\bar{v} \in F$, $L > 0$, $\eta > 0$, $\nu > 0$, and $\lambda_0 \in \mathbb{R}^N$ be as defined in the proof of Thorem A.1 in Section A.3. Choose $\kappa > 0$ such that $\lambda_0 \cdot (g(a) - \bar{v}) < -\kappa$ for any $a \notin g^{-1}(\bar{v})$, and $\theta \in (0,1)$ so that $1-\theta^2 < (\kappa/2L)^3$.

Claim A.1. *For each $v' \in F$, we have*

$$\lambda_0 \cdot (v' - \bar{v}) \leq -\frac{\kappa}{L}\|v' - \bar{v}\|. \tag{9}$$

Proof. Take any $v' \in F \setminus \{\bar{v}\}$. Let $\beta \in \Delta(A)$ be such that $v' = \sum_{a\in A} \beta(a)g(a)$. Note that

$$\begin{aligned}\lambda_0 \cdot (v' - \bar{v}) &= \sum_{a\notin g^{-1}(\bar{v})} \beta(a)\lambda_0 \cdot (g(a) - \bar{v}) \\ &\leq -\kappa \sum_{a\notin g^{-1}(\bar{v})} \beta(a).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|v' - \bar{v}\| &\leq \sum_{a\notin g^{-1}(\bar{v})} \beta(a)\|g(a) - \bar{v}\| \\ &\leq L \sum_{a\notin g^{-1}(\bar{v})} \beta(a).\end{aligned}$$

Combining the above two inequalities, we obtain (9). □

To prove the lemmas, I define two functions $\underline{q}, \bar{q} \colon F \setminus \{\bar{v}\} \to \mathbb{R}_{++}$ by

$$\underline{q}(v') = \frac{-\lambda_0 \cdot (v' - \bar{v}) - \sqrt{(\lambda_0 \cdot (v' - \bar{v}))^2 - (1-\theta)^2\|v' - \bar{v}\|^2}}{1-\theta^2},$$

$$\bar{q}(v') = \frac{-\lambda_0 \cdot (v' - \bar{v}) + \sqrt{(\lambda_0 \cdot (v' - \bar{v}))^2 - (1-\theta)^2 \|v' - \bar{v}\|^2}}{1 - \theta^2}$$

for each $v' \in F \setminus \{\bar{v}\}$. Note that $0 < \underline{q}(v') < \bar{q}(v')$ for any $v' \in F \setminus \{\bar{v}\}$, since $(\lambda_0 \cdot (v' - \bar{v}))^2 - (1-\theta)^2 \|v' - \bar{v}\|^2 > 0$ by the assumption that $1 - \theta^2 < (\kappa/2L)^3$ and Claim A.1. I define $\underline{q}$ and $\bar{q}$ so that $v' \in B_q$ if and only if $\underline{q}(v') \leq q \leq \bar{q}(v')$ for any $v' \in F \setminus \{\bar{v}\}$ and $q > 0$. See Fig. 3 for the graphical meanings of $\underline{q}$ and $\bar{q}$. We will use the following properties of $\underline{q}$ and $\bar{q}$.

Claim A.2. *For each $v' \in F \setminus \{\bar{v}\}$, $\underline{q}(v')$ and $\bar{q}(v')$ satisfy the following:*

(i) $0 < \underline{q}(v') < \bar{q}(v')$.

(ii) For each $q >$, $v' \in B_q$ if and only if $\underline{q}(v') \leq q \leq \bar{q}(v')$.

(iii) $\|v' - \bar{v}\| \geq (\kappa/L)\underline{q}(v')$.

(iv) $((1-\theta^2)/2)\bar{q}(v') \leq \|v - \bar{v}\| \leq (L(1-\theta^2)/\kappa)\bar{q}(v')$.

In particular, (ii) implies $v' \in S_{\bar{q}(v')}$, that is,

$$\|v' - \bar{v} + \bar{q}(v')\| = \theta \bar{q}(v').$$

Proof. Fix any $v' \in F \setminus \{\bar{v}\}$. Note that for each $q > 0$, $v' \in B_q$ if and only if

$$\|v' - \bar{v} + \bar{q}(v')\| \leq \theta \bar{q}(v')$$

or

$$(1-\theta^2)q^2 + 2q\lambda_0 \cdot (v' - \bar{v}) + \|v' - \bar{v}\|^2 \leq 0. \tag{10}$$

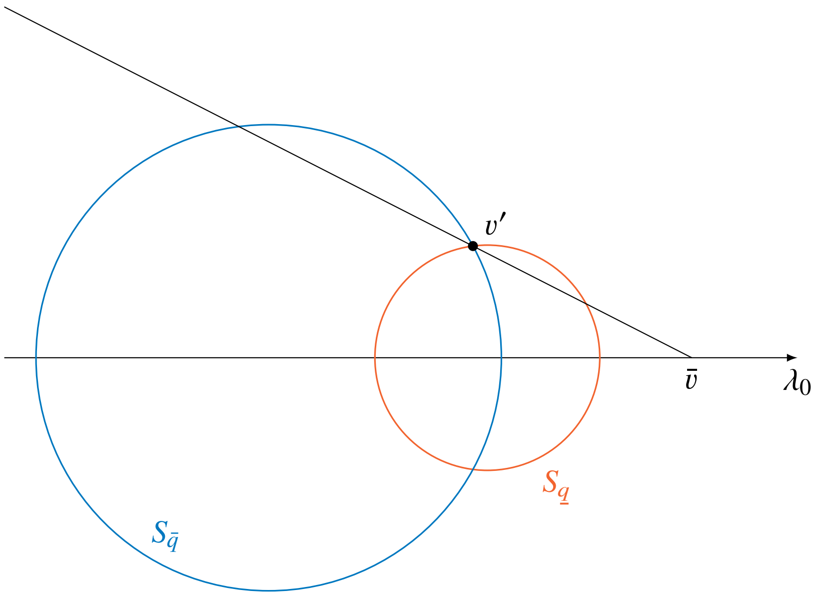
Since $\lambda_0 \cdot (v' - \bar{v}) \leq -(\kappa/L)\|v' - \bar{v}\|$ (Claim A.1) and $1 - \theta^2 < 3(\kappa/2L)^2$, we have

$$(\lambda_0 \cdot (v' - \bar{v}))^2 - (1-\theta^2)\|v' - \bar{v}\|^2 \geq \left(\frac{\kappa}{2L}\right)^2 \|v' - \bar{v}\|^2 > 0. \tag{11}$$

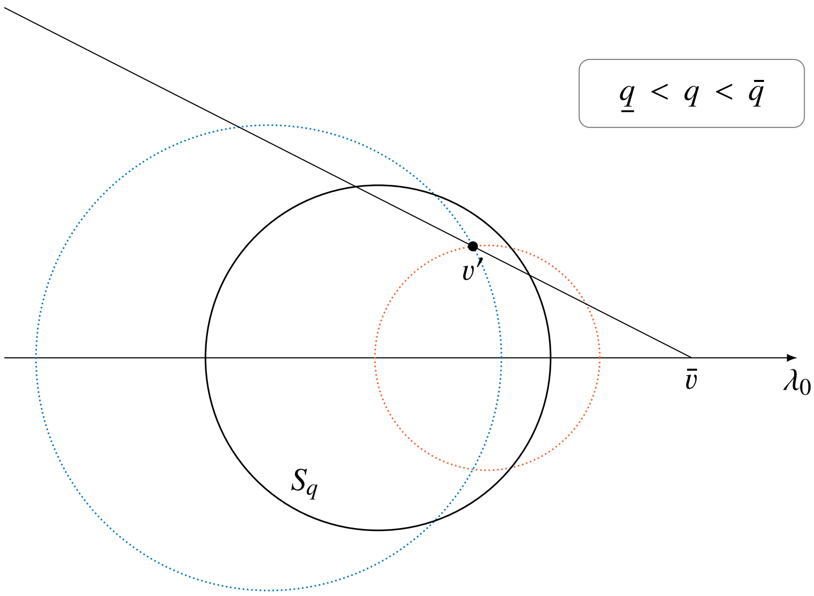
Hence, $0 < \underline{q}(v') < \bar{q}(v')$, and (10) is equivalent to $q \in [\underline{q}(v'), \bar{q}(v')]$. We have proved (i) and (ii). One can show (iii) and (iv) by substituting inequality (11) into the expressions for $\underline{q}(v')$ and $\bar{q}(v')$. □

By using $\bar{q}$, we can characterize F_q, $q(\delta)$, and v in the following way.

- $F_q = \{v' \in F \setminus \{\bar{v}\} \mid \bar{q}(v) \geq q\}$ for each $q > 0$,
- $q(\delta) = \min_{v' \in E(\delta)} \bar{q}(v')$ for each $\delta < 1$, and



(a) For a payoff vector $v' \in F \setminus \{\bar{v}\}$, let $\underline{q} := \underline{q}(v')$, and $\bar{q} := \bar{q}(v')$. The sphere $S_{\underline{q}}$ crosses the ray $\{\bar{v} + t(v' - \bar{v} \mid t \geq 0\}$ at two points with v' as the "right" intersection, while $S_{\bar{q}}$ crosses the ray with v' as the "left" intersection.



(b) If $\underline{q} < q < \bar{q}$, then the payoff vector v' lies in the interior of the closed ball B_q. The circle drawn with a thick line depicts the sphere S_q of B_q.

Fig. 3: The graphical meanings of $\underline{q}$ and $\bar{q}$.

- $v \in E(\delta)$ is any equilibrium payoff vector such that $\bar{q}(v) = q(\delta)$.

Proof of Lemma A.2. Take any $v' \in F_{q(\delta)}$. It suffices to show that

$$(v - \bar{v} + q(\delta)\lambda_0) \cdot v' \leq (v - \bar{v} + q(\delta)\lambda_0) \cdot v. \tag{12}$$

Decompose v' as

$$v' = \left(\bar{v} + \frac{q(\delta)}{\bar{q}(v')}(v' - \bar{v})\right) + \left(v' - \bar{v} - \frac{q(\delta)}{\bar{q}(v')}(v' - \bar{v})\right),$$

where the first term on the right-hand side represents the one closer to $\bar{v}$ among the two points at which the sphere $S_{q(\delta)}$ crosses the ray $\{\bar{v} + t(v' - \bar{v}) \mid t \geq 0\}$. For the first term,

$$\begin{aligned}
&(v - \bar{v} + \bar{q}(v)\lambda_0) \cdot \left(\bar{v} + \frac{q(\delta)}{\bar{q}(v')}(v' - \bar{v})\right) - (v - \bar{v} + q(\delta)\lambda_0) \cdot v \\
=&(v - \bar{v} + \bar{q}(v)\lambda_0) \cdot \left(\bar{v} + \frac{q(\delta)}{\bar{q}(v')}(v' - \bar{v}) - \bar{v} + q(\delta)\lambda_0\right) - (v - \bar{v} + q(\delta)\lambda_0) \cdot (v - \bar{v} + q(\delta)\lambda_0) \\
=&\frac{q(\delta)}{\bar{q}(v')}(v - \bar{v} + q(\delta)\lambda_0) \cdot (v' - \bar{v} + \bar{q}(v')\lambda_0) - \|v - \bar{v} + q(\delta)\lambda_0\|^2 \\
\leq&\frac{q(\delta)}{\bar{q}(v')}\|v - \bar{v} + q(\delta)\lambda_0\| \cdot \|v' - \bar{v} + \bar{q}(v')\lambda_0\| - \|v - \bar{v} + q(\delta)\lambda_0\|^2 \\
=&\frac{q(\delta)}{\bar{q}(v')}(\theta q(\delta)) \cdot (\theta q(v')) - (\theta q(\delta))^2 \\
=&0.
\end{aligned}$$

For the second term,

$$\begin{aligned}
&(v - \bar{v} + q(\delta)\lambda_0) \cdot \left(v' - \bar{v} - \frac{q(\delta)}{\bar{q}(v')}(v' - \bar{v})\right) \\
=&\left(1 - \frac{q(\delta)}{\bar{q}(v')}\right)(v - \bar{v} + q(\delta)\lambda_0) \cdot (v' - \bar{v}) \\
=&\left(1 - \frac{q(\delta)}{\bar{q}(v')}\right)((v - \bar{v}) \cdot (v' - \bar{v}) + q(\delta)\lambda_0 \cdot (v' - \bar{v})) \\
\leq&\left(1 - \frac{q(\delta)}{\bar{q}(v')}\right)\left(\|v - \bar{v}\| \cdot \|v' - \bar{v}\| - \frac{\kappa}{L}q(\delta)\|v' - \bar{v}\|\right) \\
\leq&\left(1 - \frac{q(\delta)}{\bar{q}(v')}\right)\|v' - \bar{v}\|\left(\frac{L}{\kappa}(1 - \theta^2)q(\delta) - \frac{\kappa}{L}q(\delta)\right) \\
\leq&\left(1 - \frac{q(\delta)}{\bar{q}(v')}\right)\|v' - \bar{v}\|q(\delta)\frac{L}{\kappa}\left(1 - \theta^2 - \left(\frac{\kappa}{L}\right)^2\right)
\end{aligned}$$

$$\leq 0,$$

where we have used Claim A.1, property (iv) in Claim A.2, and the fact that $\bar{q}(v') \geq q(\delta)$ and $1-\theta^2 < (\kappa/L)^2$. Therefore, we obtain (12). □

Proof of Lemma A.4. Take any $\bar{\xi} > 0$. By Lemma A.1, there exists $\chi_0 > 0$ such that for any $\alpha \in \times_i \Delta(A_i)$ with $\sum_{a \in g^{-1}(\bar{v})} \alpha(a) > 1 - \chi_0$, there exist $\xi \in [0, \bar{\xi})$, $\bar{\alpha} \in \times_i \Delta(A_i)$, and $\underline{\alpha} \in \Delta(A)$ such that $\alpha = (1-\xi)\bar{\alpha} + \xi\underline{\alpha}$ and $\operatorname{supp} \bar{\alpha} \subset g^{-1}(\bar{v})$. Let $\chi := \frac{\kappa^2 \chi_0}{L(1-\theta^2)}$.

Suppose that $q(\delta) < \chi$. By Lemma A.2,

$$\begin{aligned}
\lambda \cdot v &= (1-\delta)\lambda \cdot g(\alpha) + \delta \sum_{y \in Y} \pi(y \mid \alpha) \lambda \cdot w(y) \\
&\leq (1-\delta)\lambda \cdot g(\alpha) + \delta \lambda \cdot v.
\end{aligned}$$

We thus have

$$\begin{aligned}
\lambda \cdot (g(\alpha) - \bar{v}) &\geq \lambda \cdot (v - \bar{v}) \\
&\geq -\|v - \bar{v}\| \\
&\geq -\frac{L}{\kappa}(1-\theta^2) q(\delta),
\end{aligned}$$

where the inequality in the last line follows from property (iv) in Claim A.2. On the other hand,

$$\begin{aligned}
\lambda \cdot (g(\alpha) - \bar{v}) &= \sum_{a \in g^{-1}(\bar{v})} \lambda \cdot (g(\alpha) - \bar{v}) \\
&\leq -\kappa \sum_{a \notin g^{-1}(\bar{v})} \alpha(a).
\end{aligned}$$

Hence, we have

$$\sum_{a \notin g^{-1}(\bar{v})} \alpha(a) \leq \frac{L}{\kappa^2}(1-\theta^2) q(\delta) < \chi_0.$$

Therefore, there exist $\xi \in [0, \bar{\xi})$, $\bar{\alpha} \in \times_i \Delta(A_i)$, and $\underline{\alpha} \in \Delta(A)$ such that $\alpha = (1-\xi)\bar{\alpha} + \xi\underline{\alpha}$ and $g(\bar{\alpha}) = \bar{v}$.

□

Claim A.3.

$$\|\lambda - \lambda_0\| \leq \frac{4L}{\kappa}(1-\theta^2).$$

Proof. Note that

$$\begin{aligned}\lambda - \lambda_0 &= \frac{v - \bar{v} + q(\delta)\lambda_0}{\theta q(\delta)} - \lambda_0 \\ &= \frac{1}{\theta q(\delta)}(v - \bar{v}) + \frac{1-\theta}{\theta}\lambda_0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\lambda - \lambda_0\| &\leq \frac{1}{\theta q(\delta)}\|v - \bar{v}\| + \frac{1-\theta}{\theta} \\ &\leq \frac{L}{\kappa} \cdot \frac{1-\theta^2}{\theta} + \frac{1-\theta}{\theta} \\ &\leq \frac{2L}{\kappa} \cdot \frac{1-\theta^2}{\theta} \\ &\leq \frac{4L}{\kappa}(1-\theta^2),\end{aligned}$$

where we have used property (iv) in Claim A.2 and the fact that $\theta > 1/2$. □

We will prove the following claim that is stronger than Lemma A.3.

Claim A.4. *For each $v' \in F$, we have*

$$\lambda \cdot (v' - \bar{v}) \leq -\frac{\kappa}{2L}\|v' - \bar{v}\|,$$

in particular

$$\lambda \cdot v' \leq \lambda \cdot \bar{v}.$$

Proof. Take any $v' \in F$.

$$\begin{aligned}\lambda \cdot (v' - \bar{v}) &= \lambda_0 \cdot (v' - \bar{v}) + (\lambda - \lambda_0) \cdot (v' - \bar{v}) \\ &\leq -\frac{\kappa}{L}\|v' - \bar{v}\| + \|\lambda - \lambda_0\| \cdot \|v' - \bar{v}\| \\ &\leq -\frac{\kappa}{L}\|v' - \bar{v}\| + \frac{4L}{\kappa}(1-\theta^2)\|v' - \bar{v}\| \\ &= \frac{4L}{\kappa}\|v' - \bar{v}\|\left((1-\theta^2) - \left(\frac{\kappa}{2L}\right)^2\right) \\ &\leq -\frac{\kappa}{2L}\|v' - \bar{v}\|,\end{aligned}$$

where we have used Claims A.1, A.3, and the fact that $1 - \theta^2 < (1/2)(\kappa/2L)^2$. □

Proof of Lemma A.3. Lemma A.3 is an immediate corollary of Claim A.3. □

Claim A.5. *For any $v' \in F \setminus \{\bar{v}\}$ with $\|v' - \bar{v}\| \leq (\kappa/L)q$, we have $q \geq \underline{q}(v')$.*

Proof. Immediate by property (iii) in Claim A.2. □

Proof of Lemma A.5. Take any $y \in Y$. By property (ii) in Claim A.2, it suffices to show that $q(\delta) \in [\underline{q}(w(y)), \bar{q}(w(y))]$. Since $w(y) \in E(\delta) \subset F_{q(\delta)}$, we have $\bar{q}(w(y)) \geq q(\delta)$. Note that property (iii) in Claim A.2 implies that for any $q > 0$ and $v' \in F \setminus \{\bar{v}\}$ with $\|v' - \bar{v}\| \leq (\kappa/L)q$, we have $q \geq \underline{q}(v')$. We will thus show that $\|w(y) - \bar{v}\| \leq (\kappa/L)q(\delta)$. By Claim A.4,

$$\lambda \cdot (w(y) - \bar{v}) \leq -\frac{\kappa}{2L}\|w(y) - \bar{v}\|.$$

On the other hand,

$$\begin{aligned}
\lambda \cdot (w(y) - \bar{v}) &= \lambda \cdot (w(y) - v) + \lambda \cdot (v - \bar{v}) \\
&\geq -\frac{2}{\delta\eta}(1-\delta)\lambda \cdot (v - \bar{v}) + \lambda.(v - \bar{v}) \\
&\geq \left(1 - \frac{2}{\delta\eta}(1-\delta)\right)\lambda \cdot (v - \bar{v}) \\
&= \frac{1}{2}\lambda \cdot (v - \bar{v}) \\
&> -\frac{L(1-\theta^2)}{2\kappa}q(\delta),
\end{aligned}$$

where we have used inequality (8), property (iv) in Claim A.2, and the fact that δ is sufficiently close to 1 so that $2(1-\delta)/\delta\eta < 1/2$. Hence,

$$\|w(y) - \bar{v}\| \leq \left(\frac{2L}{\kappa}\right)^2 (1-\theta^2)q(\delta).$$

Therefore,

$$\begin{aligned}
\|w(y) - \bar{v}\| - \frac{\kappa}{L}q(\delta) &\leq \left(\left(\frac{2L}{\kappa}\right)^2 (1-\theta^2) - \frac{\kappa}{L}\right)q(\delta) \\
&= \left(\frac{2L}{\kappa}\right)^2 \left((1-\theta)^2 - 2\left(\frac{\kappa}{2L}\right)^3\right)q(\delta) \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the fact that $1 - \theta^2 < (\kappa/2L)^3$. □

B Proof of Proposition 1

In this section, I will prove Proposition 1 relying on Theorem A.1 (or Corollary A.2). First, I will prove the following preliminary lemma.

Lemma B.1. *Let $\bar{v} := (\overline{W}, \overline{W})$. For each feasible payoff vector $v = (v_1, v_2) \in F$,*

$$\overline{W} - \frac{v_1 + v_2}{2} \geq \frac{\gamma - 1}{4(\gamma + 1)} \|v - \bar{v}\|.$$

Proof. Let $\mathbb{1} = (1, 1)$. For each $a \notin g^{-1}(\bar{v})$,

$$\mathbb{1} \cdot (g(a) - \bar{v}) \leq -p(\gamma - 1),$$

and

$$\|g(a) - \bar{v})\| \leq 2p(\gamma + 1).$$

Take any $v \in F \setminus \{\bar{v}\}$. Let $\beta \in \Delta(A)$ be such that $v = \sum_{a \in A} \beta(a) g(a)$. Note that

$$\begin{aligned} \mathbb{1} \cdot (v - \bar{v}) &= \sum_{a \notin g^{-1}(\bar{v})} \beta(a) \mathbb{1} \cdot (g(a) - \bar{v}) \\ &\leq -p(\gamma - 1) \sum_{a \notin g^{-1}(\bar{v})} \beta(a), \end{aligned}$$

and

$$\begin{aligned} \|v - \bar{v}\| &\leq \sum_{a \notin g^{-1}(\bar{v})} \beta(a) \|g(a) - \bar{v}\| \\ &\leq 2p(\gamma + 1) \sum_{a \notin g^{-1}(\bar{v})} \beta(a). \end{aligned}$$

Thus, we have

$$\mathbb{1} \cdot (v - \bar{v}) \leq -\frac{\gamma - 1}{2(\gamma + 1)} \|v - \bar{v}\|.$$

Rearranging the formula, we obtain

$$\overline{W} - \frac{v_1 + v_2}{2} \geq \frac{\gamma - 1}{4(\gamma + 1)} \|v - \bar{v}\|.$$

□

Proof of Proposition 1. Let $\bar{v} = (\overline{W}, \overline{W})$. By Corollary A.2,

$$d\big(\bar{v}, E^{\mathrm{f}}(\delta)\big) = \Omega\big((1 - \delta)^{1/2}\big).$$

By Lemma B.1,

$$\begin{aligned}\overline{W} - W^*(\delta) &= \inf_{(v_1,v_2)\in E^{\mathrm{f}}(\delta)} \left(\overline{W} - \frac{v_1+v_2}{2}\right) \\ &\geq \frac{\gamma-1}{4(\gamma+1)} \inf_{v\in E^{\mathrm{f}}(\delta)} \|v-\bar{v}\| \\ &= \frac{\gamma-1}{4(\gamma+1)} d\big(\bar{v}, E^{\mathrm{f}}(\delta)\big).\end{aligned}$$

Therefore,

$$\overline{W} - W^*(\delta) = \Omega\big((1-\delta)^{1/2}\big) \quad \text{as} \quad \delta \to 1.$$

□

C Proof of Proposition 2

C.1 Derivation of the Value Function

Fix $\delta < 1$ and $n \in \mathbb{N}$. Hereafter I will suppress the dependence of variables on the discount factor δ and the number n of chips when there is no confusion. The value function satisfies the following system of equations:

$$V_k = \begin{cases} p\delta V_1 + (1-p)\delta V_0 & \text{if } k = 0 \\ p\left((1-\delta) + \delta V_{2n}\right) + p\left((1-\delta)\gamma + \delta V_{2n-1}\right) + (1-2p)\delta V_{2n} & \text{if } k = 2n \\ p\delta V_{k+1} + p\left((1-\delta)\gamma + \delta V_{k-1}\right) + (1-2p)\delta V_k & \text{otherwise.} \end{cases} \tag{13}$$

Let

$$\zeta = \zeta(\delta) := \frac{1-\delta}{p\delta},$$

and

$$\Delta_k = \Delta_k^n(\delta) := \frac{\delta\left(V_{k+1}^n(\delta) - V_k^n(\delta)\right)}{1-\delta}$$

for each $k \in \{0, \ldots, 2n-1\}$. Rearranging the equations (13), we obtain

$$V_k = \begin{cases} p\Delta_0 & \text{if } k = 0 \\ p\gamma - p(\Delta_{2n-1} - 1) & \text{if } k = 2n \\ p\gamma - p(\Delta_{k-1} - \Delta_k) & \text{otherwise,} \end{cases} \tag{14}$$

and

$$(2+\zeta)\Delta_0 - \Delta_1 = \gamma, \tag{15}$$

$$-\Delta_{2n-2} + (2+\zeta)\Delta_{2n-1} = 1, \tag{16}$$

$$-\Delta_{k-1} + (2+\zeta)\Delta_k - \Delta_{k+1} = 0 \quad (k \in \{1,\dots,2n-2\}). \tag{17}$$

Since equations (17) are second-order difference equations, we can write

$$\Delta_k = c_1\alpha^k + c_2\beta^k \quad (k \in \{0,\dots,2n-1\}) \tag{18}$$

for some $c_1, c_2 \in \mathbb{R}$, where

$$\alpha = \frac{2+\zeta-\sqrt{(2+\zeta)^2-4}}{2},$$
$$\beta = \frac{2+\zeta+\sqrt{(2+\zeta)^2-4}}{2}.$$

Note that $0 < \alpha < 1 < \beta$. By using equations (15), (16), and (18), we can get c_1 and c_2

$$\begin{aligned}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} \beta & \alpha \\ \alpha^{2n} & \beta^{2n} \end{pmatrix}^{-1} \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \\ &= \frac{1}{\beta^{2n+1}-\alpha^{2n+1}} \begin{pmatrix} \beta^{2n}\gamma - \alpha \\ -(\alpha^{2n}\gamma - \beta) \end{pmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_k &= \frac{(\beta^{2n}\gamma-\alpha)\alpha^k - (\alpha^{2n}\gamma-\beta)\beta^k}{\beta^{2n+1}-\alpha^{2n+1}} \\ &= \frac{\gamma(\alpha^k\beta^{2n}-\alpha^{2n}\beta^k) + (\beta^{k+1}-\alpha^{k+1})}{\beta^{2n+1}-\alpha^{2n+1}}\end{aligned} \tag{19}$$

for $k \in \{0,\dots,2n-1\}$. Note that $\Delta_k > 0$ for all $k \in \{0,\dots,2n-1\}$, which implies that V_k is strictly increasing in k. The value V_k of holding k chips is obtained by substituting (19) into (14). In particular,

$$W_{\mathrm{c}}^n(\delta) = V_n^n(\delta) = \overline{W} - p(\gamma-1)\frac{\alpha^n(1-\alpha)}{1-\alpha^{2n+1}}, \tag{20}$$

where I use the fact that $\beta = 1/\alpha$. Moreover, for each $\delta \in (0, 1)$,

$$W_c^*(\delta) = W_c^{n^*(\delta)}(\delta) \tag{21}$$

since $W_c^n(\delta)$ is (strictly) increasing in n. It follows from (20) and (21) that

$$\overline{W} - W_c^*(\delta) = p(\gamma - 1)\frac{\alpha^{n^*(\delta)}(1-\alpha)}{1-\alpha^{2n^*(\delta)+1}}. \tag{22}$$

C.2 Convergence Rate of $\overline{W} - W_c^*(\delta)$

Recall that the n-chip strategies constitute an equilibrium if and only if

$$\delta V_{k+1} \geq (1-\delta) + \delta V_k \tag{23}$$

for all $k \in \{0, \ldots, 2n-1\}$. Incentive condition (23) can be expressed by Δ_k's;

$$\Delta_k \geq 1 \quad (k \in \{0, \ldots, 2n-1\}). \tag{24}$$

Abdulkadiroglu and Bagwell (2012) show that the only binding incentive condition is the one for the player with $2n-1$ chips. That is, the n-chip strategies constitute an equilibrium if and only if $\Delta_{2n-1} \geq 1$. I will state its proof in Lemmas C.1 and C.2 for self-containedness.[16] Lemma C.1 and C.2 below correspond to their Lemma 4 and 6 in Abdulkadiroglu and Bagwell (2012), respectively.

Lemma C.1. *For any $n \in \mathbb{N}$ and $\delta < 1$, if $\Delta_{2n-1}^n(\delta) \geq 1$ then $V_k^n(\delta) \leq p\gamma$ for all $k \in \{0, \ldots, 2n\}$.*

Proof. Fix any $n \in \mathbb{N}$ and $\delta < 1$. Suppose that $\Delta_{2n-1} \geq 1$. By equation (14) for $k = 2n$, we have

$$p\gamma - V_{2n} = p(\Delta_{2n-1} - 1) \geq 0,$$

or equivalently $V_{2n} \leq p\gamma$. Since $\Delta_k > 0$ for all $k \in \{0, \ldots, 2n-1\}$, we obtain $V_k \leq p\gamma$ for all $k \in \{0, \ldots, 2n\}$. □

Lemma C.2. *For any $n \in \mathbb{N}$ and $\delta < 1$, the n-chip strategies constitute an equilibrium if and only if $\Delta_{2n-1}^n(\delta) \geq 1$.*

Proof. Fix any $n \in \mathbb{N}$ and $\delta \in (0, 1)$. To prove the sufficiency, suppose that $\Delta_{2n-1} \geq 1$. By Lemma C.1, $V_k \leq p\gamma$ for all $k \in \{0, \ldots, 2n\}$. Hence equations (14) for $k \neq 0, 2n$

[16] The proofs of Lemmas C.1 and C.2 are shorter than those of Abdulkadiroglu and Bagwell (2012), since I employ the formulas derived in Section C.1.

imply that Δ_k is nonincreasing in k. Thus we have $\Delta_k \geq \Delta_{2n-1} \geq 1$ for all $k \in \{0, \ldots, 2n-1\}$. That is, the n-chip strategies constitute an equilibrium. □

Now, I will derive the divergence rate of $n^*(\delta)$, the maximum number of chips which can be supported in chip-strategy equilibria when the discount factor is δ. As we have seen in Section C.1, $\Delta^n_{2n-1}(\delta)$ depends on δ only through ζ. Hence, for simplicity, I will treat ζ instead of δ as an independent variable from now on. For each $n \in \mathbb{N}$, let ζ^*_n denote the largest $\zeta > 0$ for which the n-chip strategies constitute an equilibrium.[17,18] To derive the divergence rate of $n^*(\delta)$, I consider another problem; what is the convergence rate of ζ^*_n as the number n of chips grows to infinity.

Lemma C.3. *ζ^*_n vanishes at rate at least $1/n^2$. That is,*

$$\zeta^*_n = \Omega\left(\frac{1}{n^2}\right). \tag{25}$$

Proof. Fix $n \in \mathbb{N}$ and take any $\zeta > 0$. By Lemma C.2, the n-chip strategies constitute an equilibrium if and only if

$$\Delta_{2n-1} \geq 1. \tag{26}$$

Note that condition (26) is equivalent to

$$-\alpha^{4n+1} + \gamma\alpha^{2n}(1+\alpha) - 1 \geq 0. \tag{27}$$

Define a function $\phi_n \colon [0,1] \to \mathbb{R}$ by

$$\phi_n(t) = -(1-t)^{4n} + \gamma(1-t)^{2n}(2-t) - 1$$

for each $t \in [0,1]$. Function ϕ_n is continuous and strictly decreasing. Moreover, we have $\phi_n(0) = 2(\gamma - 1) > 0$ and $\phi_n(1) = -1 < 0$. Hence, by the Intermediate Value Theorem there exists a unique solution to the equation $\phi_n(t) = 0$, $t \in (0,1)$. Let $t_n \in (0,1)$ denote the solution. The n-chip strategies constitute an equilibrium if and only if

$$\alpha \geq 1 - t_n,$$

or equivalently

$$\zeta \leq \frac{t_n^2}{1-t_n}.$$

[17] ζ_{n^*} is well defined, since $\Delta^n_{2n-1}(\delta)$ is a continuous function of δ, and hence of ζ.

[18] Recall that $\zeta(\delta) = p(1-\delta)/\delta$ and hence a low ζ associates with a high δ.

Therefore,

$$\zeta_n = \frac{t_n^2}{1 - t_n}.$$

Since $\zeta_n \geq t_n^2$, it suffices to show that

$$t_n = \Omega\left(\frac{1}{n}\right). \tag{28}$$

Suppose for a contradiction that

$$t_n \neq \Omega\left(\frac{1}{n}\right).$$

We can construct a sequence $\{n_l\}$ of positive integers such that $n_l \to \infty$ and $n_l t_{n_l} \to 0$ as $l \to \infty$. Let $\nu > 0$ be so small that $-e^{4\nu} + 2\gamma e^{-2\nu} - 1 > 0$. For a sufficiently large $l \in \mathbb{N}$, we have

$$\begin{aligned} 0 &= \phi_{n_l}(t_{n_l}) \\ &\geq \phi_{n_l}(\nu/n_l) \\ &= -\left(1 - \frac{\nu}{n_l}\right)^{4n_l} + \gamma\left(1 - \frac{\nu}{n_l}\right)^{2n_l}\left(2 - \frac{\nu}{n_l}\right) - 1, \end{aligned}$$

where the inequality on the second line comes from the fact that $n_l t_{n_l} \leq \nu$ for a sufficiently large $l \in \mathbb{N}$. The last expression converges to $-e^{4\nu} + 2\gamma e^{-2\nu} - 1 > 0$ as $l \to \infty$. This is a contradiction. Hence we must have (28), so (25).

□

Now, we can derive a lower bound on the rate of divergence for $n^*(\delta)$ by reversing the roles of n and ζ (or δ).

Lemma C.4.

$$n^*(\delta) = \Omega\left(\frac{1}{(1-\delta)^{1/2}}\right).$$

Proof. Since $\zeta_n > 0$ for all $n \in \mathbb{N}$ and $\zeta(\delta) \to 0$ as $\delta \to 1$, we have $n^*(\delta) \to \infty$ as $\delta \to 1$. Moreover, by Lemma C.3, there exist $c > 0$ and $N \in \mathbb{N}$ such that $\zeta_n > c/n^2$ for all $n \geq N$. Let $\underline{\delta} < 1$ be such that $n^*(\delta) \geq N$ for all $\delta > \underline{\delta}$. Take any $\delta > \underline{\delta}$. We have

$$\zeta_{n^*(\delta)+1} > \frac{c}{(n^*(\delta)+1)^2}.$$

Moreover, by the definition of n^*, we must have

$$\zeta(\delta) > \zeta_{n^*(\delta)+1}.$$

Hence we have

$$\zeta(\delta) > \frac{c}{(n^*(\delta)+1)^2}.$$

By rearranging the above inequality, we obtain

$$n^*(\delta)\zeta(\delta)^{1/2} > c^{1/2} - \zeta(\delta)^{1/2}.$$

Therefore,

$$n^*(\delta) = \Omega\left(\zeta^{-1/2}\right),$$

or equivalently

$$n^*(\delta) = \Omega\left(\frac{1}{(1-\delta)^{1/2}}\right).$$

□

We are now in a position to prove Proposition 2.

Proof of Proposition 2. By Lemma C.4, there exist a positive constant $c > 0$ and $\underline{\delta} < 1$ such that

$$n^*(\delta) > c(1-\delta)^{-1/2} \tag{29}$$

for all $\delta > \underline{\delta}$. Take any $\delta > \underline{\delta}$. For simplicity, let $n^* = n^*(\delta)$. By (22), we have

$$\begin{aligned}\overline{W} - W_c^*(\delta) &\leq p(\gamma-1)\frac{1-\alpha}{1-\alpha^{2n^*+1}} \\ &\leq p(\gamma-1)(1-\alpha)\left(1-\alpha^{2c(1-\delta)^{-1/2}+1}\right)^{-1}.\end{aligned}$$

One can easily check

$$\begin{aligned}&\lim_{\delta\to 1}(1-\delta)^{-1/2}(1-\alpha(\delta)) = 1/\sqrt{p}, \\ &\lim_{\delta\to 1}\alpha(\delta)^{2c(1-\delta)^{-1/2}+1} = e^{-2c/\sqrt{p}} < 1.\end{aligned}$$

Therefore, we obtain

$$\overline{W} - W_c^*(\delta) = O\left((1-\delta)^{1/2}\right).$$

□

D Coefficients for $n^*(\delta)$ and $\overline{W} - W_c^*(\delta)$

I obtain the leading coefficients for $n^*(\delta)$ and $\overline{W} - W_c^*(\delta)$ by refining the proof of Proposition 2.

Proposition D.1.
$$\lim_{\delta \to 1} n^*(\delta)(1-\delta)^{1/2} = c_{p,\gamma} \tag{30}$$
and
$$\lim_{\delta \to 1} \left(\overline{W} - W_c^*(\delta)\right)(1-\delta)^{-1/2} = C_{p,\gamma}, \tag{31}$$
where
$$c_{p,\gamma} = \frac{\sqrt{p}}{2} \log\left(\gamma + \sqrt{\gamma^2 - 1}\right),$$
$$C_{p,\gamma} = \left(\frac{p(\gamma - 1)}{2}\right)^{1/2}.$$

I prove Proposition D.1 by refining Lemma C.3.

Lemma D.1.
$$\lim_n n^2 \zeta_n = \nu_\gamma^2, \tag{32}$$
where
$$\nu_\gamma = \frac{1}{2}\log\left(\gamma + \sqrt{\gamma^2 - 1}\right) = c_{p,\gamma}/\sqrt{p}.$$

Proof. For each $n \in \mathbb{N}$, let ϕ_n and t_n be as defined in the proof of Lemma C.3. Since $t_n \to 0$ as $n \to \infty$, it suffices to show that
$$\lim_n n t_n = \nu_\gamma. \tag{33}$$

Define a function $\psi \colon \mathbb{R}_+ \to \mathbb{R}$ by
$$\psi(\nu) = -e^{-4\nu} + 2\gamma e^{-2\nu} - 1$$
for each $\nu \in \mathbb{R}_+$. Note that for all $\nu \in \mathbb{R}_+$, $\psi(\nu) > 0$ if and only if $\nu < \nu_\gamma$, and $\psi(\nu) < 0$ if and only if $\nu > \nu_\gamma$.

I show that $\limsup_n n t_n \le \nu_\gamma$ and $\liminf_n n t_n \ge \nu_\gamma$. I prove only the former, since the proofs are virtually the same. Suppose for a contradiction that $\limsup_n n t_n > \nu_\gamma$. We can find $\nu > \nu_\gamma$ and a sequence $\{n_l\}$ of positive integers such that $n_l t_{n_l} > \nu$ for all

$l \in \mathbb{N}$ and $n_l \to \infty$ as $l \to \infty$. For every $l \in \mathbb{N}$,

$$\begin{aligned} 0 &= \phi_{n_l}(t_{n_l}) \\ &\leq \phi_{n_l}(\nu/n_l) \\ &= -\left(1 - \frac{\nu}{n_l}\right)^{4n_l} + \gamma\left(1 - \frac{\nu}{n_l}\right)^{2n_l}\left(2 - \frac{\nu}{n_l}\right) - 1. \end{aligned}$$

The last expression converges to $\psi(\nu) < 0$ as $l \to \infty$. This is a contradiction. Hence, we must have $\limsup_n nt_n \leq \nu_\gamma$. Similarly, we can prove $\liminf_n nt_n \geq \nu_\gamma$. Therefore, we obtain (33), and hence (32). □

Proof of Proposition D.1. Take any $\delta < 1$. By the definition of n^*, we have

$$\zeta_{n^*(\delta)+1} < \zeta \leq \zeta_{n^*(\delta)},$$

and hence

$$\sqrt{(n^*(\delta))^2\zeta_{n^*(\delta)+1}} < n^*(\delta)\zeta^{1/2} \leq \sqrt{(n^*(\delta))^2\zeta_{n^*(\delta)}}.$$

Letting $\delta \to 1$, we obtain $\lim_{\delta\to 1} n^*(\delta)\zeta(\delta)^{1/2} = \nu_\gamma$, which implies (30).

Now, I will prove (31). Take any $\delta < 1$, and let $n^* = n^*(\delta)$. By (22), we have

$$\left(\overline{W} - W_{\mathrm{c}}^*(\delta)\right)(1-\delta)^{-1/2} = p(\gamma - 1)\frac{\alpha^{n^*}(1-\alpha)(1-\delta)^{-1/2}}{1 - \alpha^{2n^*+1}}.$$

The equation (31) follows from the following:

$$\begin{aligned} &\lim_{\delta\to 1}(1 - \alpha(\delta))(1-\delta)^{-1/2} = 1/\sqrt{p}, \\ &\lim_{\delta\to 1} \alpha(\delta)^{n^*(\delta)} = e^{-\nu_\gamma}. \end{aligned}$$

□

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